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On hazard rate ordering of the sums of heterogeneous geometric random variables

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ABSTRACT

In this paper, we treat convolutions of heterogeneous geometric random variables with respect to the p-larger order and the hazard rate order. It is shown that the p-larger order between two parameter vectors implies the hazard rate order between convolutions of two heterogeneous geometric sequences. Specially in the two-dimensional case, we present an equivalent characterization. The case when one convolution involves identically distributed variables is discussed, and we reveal the link between the hazard rate order of convolutions and the geometric mean of parameters. Finally, we drive the "best negative binomial bounds" for the hazard rate function of any convolution of geometric sequence under this setup.

1. Introduction

Because of its nice mathematical form and the memoryless property, the exponential distribution has been widely used in many areas, including life-testing, reliability, and operations research. One may refer to [1,2] for an encyclopedic treatment of developments on the exponential distribution. Convolutions of independent exponential random variables often occur naturally in many problems, and many researchers have investigated ordering properties based on exponential convolutions, including [3–8]. Also, we refer to [9–13] for ordering results of sums of independent random variables. The geometric random variable may be regarded as the discrete counterpart of the exponential random variable in the sense that both of them have constant hazard rate. In many practical situations and especially in reliability scenarios, convolutions of independent geometric random variables appear in a natural way; see [14,3] regarding some nice applications. This also motivates the present investigations. Since the distribution theory is quite complicated when the convolution involves non-identical random variables, it will be of great interest to derive bounds and approximations on some characteristics of interest in this setup.

In this paper, we investigate the ordering properties of the convolutions of heterogeneous geometric random variables. Let $X_{p_1}, \ldots, X_{p_n}$ and $X_{p_1^*}, \ldots, X_{p_n^*}$ be two sequences of independent geometric random variables with parameters $p_1, \ldots, p_n$ and $p_1^*, \ldots, p_n^*$, respectively. It is shown that for the two-dimensional case,

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$$
\begin{aligned}
& X_{p_1} + X_{p_2} \geq_{\rm hr} X_{p_1^*} + X_{p_2^*} \\
& \Longleftrightarrow X_{p_1} + X_{p_2} \geq_{\rm st} X_{p_1^*} + X_{p_2^*} \\
& \Longleftrightarrow (p_1, p_2) \overset{\rm p}{\succeq} (p_1^*, p_2^*),
\end{aligned}
$$

and for the general case,

$$(p_1, \ldots, p_n) \overset{\rm p}{\succeq} (p_1^*, \ldots, p_n^*) \Longrightarrow \sum_{i=1}^{n} X_{p_i} \geq_{\rm hr} \sum_{i=1}^{n} X_{p_i^*},$$

where the formal definitions of related partial orderings will be given in Section 2. The main results for the two-dimensional case and the general case are given in Sections 3 and 4, respectively. Specially, we discuss the case when one convolution involves identically distributed random variables and show in this case that the hazard rate order is actually associated with the geometric mean of parameters. Finally, we derive the "best negative binomial bounds" for hazard rate function of any convolution of geometric distributions.

2. Definitions

In this section, we recall some notions of stochastic orders, and majorization and related orders which are closely related to the main results to be developed in what follows. Throughout this paper, the term *increasing* is used for *monotone non-decreasing* and *decreasing* is used for *monotone non-increasing*.

Discrete stochastic orders

Definition 2.1. For two discrete random variables X and Y with common supports on integers $\mathbb{N}_0 \equiv \{0, 1, \ldots\}$, denote by $f_X(k)$ and $f_Y(k)$ their respective probability mass function (pmf), and $\overline{F}_X(k) = P(X \geq k)$ and $\overline{F}_Y(k) = P(Y \geq k)$ the corresponding survival functions. Then

(i) X is said to be smaller than Y in the likelihood ratio order, denoted by $X \leq_{\rm lr} Y$, if $f_Y(k)/f_X(k)$ is increasing in $k \in \mathbb{N}_0$;
(ii) X is said to be smaller than Y in the hazard rate order, denoted by $X \leq_{\rm hr} Y$, if $\overline{F}_Y(k)/\overline{F}_X(k)$ is increasing in $k \in \mathbb{N}_0$;
(iii) X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{\rm st} Y$, if $\overline{F}_Y(k) \geq \overline{F}_X(k)$ for all $k \in \mathbb{N}_0$.

It is well known that there exists the following implication relation:

$$X \leq_{\rm lr} Y \Longrightarrow X \leq_{\rm hr} Y \Longrightarrow X \leq_{\rm st} Y.$$

For a comprehensive discussion on various stochastic orders, one may refer to [15,16].

Majorization and related orders

The notion of majorization is quite useful in establishing various inequalities. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition 2.2. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$, written as $\mathbf{x} \overset{\rm m}{\succeq} \mathbf{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n-1,$$

and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

In addition, the vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ weakly, written as $\mathbf{x} \overset{\rm w}{\succeq} \mathbf{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$

Clearly,

$$\mathbf{x} \overset{\rm m}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\rm w}{\succeq} \mathbf{y}.$$

Those functions that preserve the majorization ordering are said to be Schur-convex. For extensive and comprehensive discussion on the theory and applications of the majorization order, one may refer to [17]. Bon and Păltănea [4] introduced a pre-order on $\mathbb{R}_+^n$, called p-larger order, which is defined as follows.

Definition 2.3. The vector $\mathbf{x}$ in $\mathbb{R}_+^n$ is said to be p-larger than another vector $\mathbf{y}$, written as $\mathbf{x} \overset{\rm p}{\succeq} \mathbf{y}$, if

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$