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Some new results on convolutions of heterogeneous gamma random variables☆

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ABSTRACT

Convolutions of independent random variables often arise in a natural way in many applied areas. In this paper, we study various stochastic orderings of convolutions of heterogeneous gamma random variables in terms of the majorization order [*p*-larger order, reciprocal majorization order] of parameter vectors and the likelihood ratio order [dispersive order, hazard rate order, star order, right spread order, mean residual life order] between convolutions of two heterogeneous gamma sets of variables wherein they have both differing scale parameters and differing shape parameters. The results established in this paper strengthen and generalize those known in the literature.

1. Introduction

Convolutions of independent random variables often arise in a natural way in many applied areas including applied probability, reliability theory, actuarial science, nonparametric goodness-of-fit testing, and operations research. Since the distribution theory is quite complicated when the convolution involves independent and non-identical random variables, it is of great interest to investigate stochastic properties of convolutions and derive bounds and approximations on some characteristics of interest in this setup. Many results in this direction have appeared in the literature; see, for example, [3,20,4,11,14,9,16,23–29,12,13,2,5]. Because exponential distribution has a nice mathematical form and the unique memoryless property, most of these references treated only the convolutions of exponential random variables. It is well known that gamma distribution is one of the most commonly used distributions in statistics, reliability and life testing that includes exponential distribution as its special case (when its shape parameter is 1). Moreover, the gamma distribution can be widely applied in actuarial science as most total insurance claim distributions have quite similar shape to gamma distributions: non-negatively supported, skewed to the right and unimodal (see [7]). Let X be a gamma random variable with the shape parameter r and scale parameter λ. Then, in its standard form X has the probability density function

$$f(x; r, \lambda) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} \exp(-\lambda x), \quad x > 0.$$

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It is an extremely flexible family of distributions with decreasing, constant, and increasing failure rates when $0 < r < 1$, $r = 1$ and $r > 1$, respectively. In this paper, various stochastic orders are studied for convolutions of heterogeneous gamma random variables.

We shall be using the concepts of majorization and related orders in our discussion. The notion of majorization is quite useful in establishing various inequalities. Let $x_{(1)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, \ldots, x_n)$.

Definition 1.1. (i) A vector $\mathbf{x} = (x_1, \ldots, x_n) \in \Re^n$ is said to majorize another vector $\mathbf{y} = (y_1, \ldots, y_n) \in \Re^n$ (written as $\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n-1,$$

and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$;

(ii) A vector $\mathbf{x} \in \Re^n$ is said to weakly supmajorize another vector $\mathbf{y} \in \Re^n$ (written as $\mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n;$$

(iii) A vector $\mathbf{x} \in \Re_+^n$ is said to be p-larger than another vector $\mathbf{y} \in \Re_+^n$ (written as $\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)} \quad \text{for } j = 1, \ldots, n.$$

Clearly, $\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y}$ implies $\mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y}$, and $\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$ is equivalent to $\log(\mathbf{x}) \overset{\text{w}}{\succeq} \log(\mathbf{y})$, where $\log(\mathbf{x})$ is the vector of logarithms of the coordinates of $\mathbf{x}$. Also, Khaledi and Kochar [8] showed that $\mathbf{x} \overset{\text{m}}{\succeq} \mathbf{y}$ implies $\mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \Re_+^n$. The converse is, however, not true. For example, $(1, 5.5) \overset{\text{p}}{\succeq} (2, 3)$, but clearly the majorization order does not hold.

For more details on majorization and p-larger orders and their applications, see [15,4,8]. Recently, Zhao and Balakrishnan [25] introduced a new partial order, called reciprocal majorization order.

Definition 1.2. The vector $\mathbf{x} \in \Re_+^n$ is said to reciprocal majorize another vector $\mathbf{y} \in \Re_+^n$ (written as $\mathbf{x} \overset{\text{rm}}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \geq \sum_{i=1}^{j} \frac{1}{y_{(i)}}$$

for $j = 1, \ldots, n$.

From [12], the following implication holds:

$$\mathbf{x} \overset{\text{w}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\text{p}}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{\text{rm}}{\succeq} \mathbf{y}$$

for any two non-negative vectors $\mathbf{x}$ and $\mathbf{y}$. On the other hand, the $\overset{\text{rm}}{\succeq}$ order does not imply the $\overset{\text{p}}{\succeq}$ order. For example, from the definition of the $\overset{\text{rm}}{\succeq}$ order, it follows that $(1, 4) \overset{\text{rm}}{\succeq} \left(\frac{4}{3}, 2\right)$, but clearly the $\overset{\text{p}}{\succeq}$ order does not hold between these two vectors.

Let us first recall some results in the literature that are most pertinent to the main results of this paper. Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$, and let $X_{\lambda_1^*}, \ldots, X_{\lambda_n^*}$ be another set of independent exponential random variables with respective hazard rates $\lambda_1^*, \ldots, \lambda_n^*$. Boland et al. [3] showed that

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{\text{lr}} \sum_{i=1}^{n} X_{\lambda_i^*}; \tag{1.1}$$

see [22,18] for a comprehensive discussion on various stochastic orders. Bon and Păltănea [4] subsequently showed that

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{p}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{\text{hr}} \sum_{i=1}^{n} X_{\lambda_i^*}, \tag{1.2}$$

and they also focused on the special case when one convolution involved identically distributed random variables. Kochar and Ma [11] established that

$$(\lambda_1, \ldots, \lambda_n) \overset{\text{m}}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{\text{disp}} \sum_{i=1}^{n} X_{\lambda_i^*}. \tag{1.3}$$