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Integral representations of one-dimensional projections for multivariate stable densities

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ABSTRACT

We consider the numerical evaluation of one-dimensional projections of general multivariate stable densities introduced by Abdul-Hamid and Nolan [H. Abdul-Hamid, J.P. Nolan, Multivariate stable densities as functions of one dimensional projections, J. Multivariate Anal. 67 (1998) 80–89]. In their approach higher order derivatives of one-dimensional densities are used, which seems to be cumbersome in practice. Furthermore there are some difficulties for even dimensions. In order to overcome these difficulties we obtain the explicit finite-interval integral representation of one-dimensional projections for all dimensions. For this purpose we utilize the imaginary part of complex integration, whose real part corresponds to the derivative of the one-dimensional inversion formula. We also give summaries on relations between various parametrizations of stable multivariate density and its one-dimensional projection.

1. Introduction

Stable distributions are known as the limiting distributions of the general central limit theorem and the time invariant distributions for Lévy processes. Due to these probabilistically very important properties, many studies have been carried out on their theoretical aspects. Comprehensive books have been published ([3] or [5]). Furthermore applications of stable distributions to heavy-tailed data have been growing over past few decades. These applications have appeared in many fields like finance, Internet traffic or physics. Especially in time series, the observations from the causal stable ARMA or fractional stable ARIMA model can be considered as a multivariate stable random vector. We refer to [17,14,15] or [2] for these applications. As various multivariate heavy-tailed data have become available in these fields, the treatments of multivariate stable distributions are needed. However, the multivariate stable distributions require handling of their spectral measures which is computationally difficult.

We briefly review some important recent progresses concerning the multivariate stable distributions. Concerning the calculation of the integral with respect to the spectral measure, Byczkowski et al. [4] have approximated the spectral measure by a discrete spectral measure which is uniformly close to the original spectral measure. They have also given several instructive graphs of the stable densities. Concerning the approximations, Davydov and Nagaev [6] have outlined possible directions although their results are still theoretical. For the estimation of the parameters, see [13,7] and for the hypothesis testing see [10]. Since there are multiple applications of these distributions, accurate calculations of the multivariate stable densities are clearly of basic importance.

For the calculations of the multivariate stable densities, the method of Abdul-Hamid and Nolan [1] based on one-dimensional projection is very promising. In the approach of Abdul-Hamid and Nolan [1], the basic ingredient is a function

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$g_{\alpha,d}(v,\beta)$ defined by

$$g_{\alpha,d}(v,\beta)=\begin{cases}\dfrac{1}{(2\pi)^d}\displaystyle\int_0^\infty \cos\left(vu-\left(\beta\tan\frac{\pi\alpha}{2}\right)u^\alpha\right)u^{d-1}\mathrm{e}^{-u^\alpha}\mathrm{d}u, & \alpha\neq 1,\\ \dfrac{1}{(2\pi)^d}\displaystyle\int_0^\infty \cos\left(vu+\frac{2}{\pi}\beta u\log u\right)u^{d-1}\mathrm{e}^{-u}\mathrm{d}u, & \alpha=1,\end{cases} \tag{1}$$

where α is the characteristic exponent of the stable distribution, d is the dimension and β corresponds to the skewness parameter.

Evaluating $g_{\alpha,d}(v,\beta)$ using this very definition is not very practical because the integrand is infinitely oscillating and changing its sign. Therefore an alternative evaluation of $g_{\alpha,d}(v,\beta)$ is desirable.

In this paper, we obtain explicit finite-interval integral representations of $g_{\alpha,d}(v,\beta)$, which do not require the derivatives of one-dimensional densities. The integrand in our representation is well behaved without infinite oscillations. Furthermore, our integral representation covers all dimensions and all parameter values, except for an exceptional case of $\alpha\le 1$, $\beta=1$ and even dimension, where we need an additional term which is easily tractable.

The calculation of $g_{\alpha,d}(v,\beta)$ corresponds to the evaluation of Eq. (2.2.18) of [19], which is stated without an explicit proof. For odd dimension, the real part of the equation is equivalent to $g_{\alpha,d}(v,\beta)$ and for even dimension, the imaginary part is equivalent to $g_{\alpha,d}(v,\beta)$. Complete derivations of Eq. (2.2.18) of [19] is also useful for the numerical evaluations of the derivatives of one-dimensional symmetric densities investigated in [9].

This paper is organized as follows. In the rest of this section we prepare preliminary results based on [1,12]. In Section 2 we derive finite-interval integral representations of one-dimensional projections. In Section 3 we summarize relations between various parametrizations of the multivariate stable density and its one-dimensional projection as propositions. Some discussions and directions for further studies are given in Section 4. Some proofs are postponed to Section 5.

1.1. Definitions and preliminary results

Let $\mathbf{X}=(X_1,X_2,\ldots,X_d)$ denote a d-dimensional α-stable random vector with the characteristic function

$$\Phi_{\alpha,d}(\mathbf{t})=E\exp(\mathrm{i}\langle\mathbf{t},\mathbf{x}\rangle),$$

where $\mathbf{t}=(t_1,t_2,\ldots,t_d)$ and $\langle\ ,\ \rangle$ denotes the usual inner product. Among several definitions of stable distributions, we give the spectral representation of $\Phi_{\alpha,d}(\mathbf{t})$. This requires integration over the unit sphere $\mathbb{S}^d=\{\mathbf{s}:\|\mathbf{s}\|=1\}$ in $\mathbb{R}^d$ with respect to a spectral measure Γ (Theorem 2.3.1 of [15], Theorem 14.10 of [16]). The spectral definition is given as

$$\Phi_{\alpha,d}(\mathbf{t})=\begin{cases}\exp\left(-\displaystyle\int_{\mathbb{S}^d}|\langle\mathbf{t},\mathbf{s}\rangle|^\alpha\left(1-\mathrm{i}\,\mathrm{sign}\,(\langle\mathbf{t},\mathbf{s}\rangle)\tan\frac{\pi\alpha}{2}\right)\Gamma(\mathrm{d}\mathbf{s})+\mathrm{i}\langle\mathbf{t},\mu\rangle\right), & \alpha\neq 1,\\ \exp\left(-\displaystyle\int_{\mathbb{S}^d}|\langle\mathbf{t},\mathbf{s}\rangle|\left(1+\mathrm{i}\frac{2}{\pi}\,\mathrm{sign}\,(\langle\mathbf{t},\mathbf{s}\rangle)\log|\langle\mathbf{t},\mathbf{s}\rangle|\right)\Gamma(\mathrm{d}\mathbf{s})+\mathrm{i}\langle\mathbf{t},\mu\rangle\right), & \alpha=1.\end{cases} \tag{2}$$

The pair (Γ,μ) is unique. Definition (2) corresponds to Zolotarev's (A) representation and we use the notation $S_{\alpha,d}(\Gamma,\mu)$ for this definition following [12].

For another definition, we give a multivariate version of Zolotarev's (M) parametrization, which is also defined in p. 191 of [12]. We use notation $S^M_{\alpha,d}(\Gamma,\mu_0)$.

$$\Phi_{\alpha,d}(\mathbf{t})=\begin{cases}\exp\left(-\displaystyle\int_{\mathbb{S}^d}|\langle\mathbf{t},\mathbf{s}\rangle|^\alpha\left(1+\mathrm{i}\,\mathrm{sign}\,(\langle\mathbf{t},\mathbf{s}\rangle)\tan\frac{\pi\alpha}{2}(|\langle\mathbf{t},\mathbf{s}\rangle|^{1-\alpha}-1)\right)\Gamma(\mathrm{d}\mathbf{s})+\mathrm{i}\langle\mathbf{t},\mu_{\mathbf{0}}\rangle\right),\\ \quad\alpha\neq 1,\\ \exp\left(-\displaystyle\int_{\mathbb{S}^d}|\langle\mathbf{t},\mathbf{s}\rangle|\left(1+\mathrm{i}\frac{2}{\pi}\,\mathrm{sign}\,(\langle\mathbf{t},\mathbf{s}\rangle)\log|\langle\mathbf{t},\mathbf{s}\rangle|\right)\Gamma(\mathrm{d}\mathbf{s})+\mathrm{i}\langle\mathbf{t},\mu_{\mathbf{0}}\rangle\right),\\ \quad\alpha=1.\end{cases} \tag{3}$$

In Eq. (3), we avoid the discontinuity at $\alpha=1$ and the divergence of the mode as $\alpha\to 1$ which are observed in definition (2) in the non-symmetric case. The difference of (M) parametrization from (A) representation is only

$$\mu_0=\mu-\tan\frac{\pi\alpha}{2}\int_{\mathbb{S}^d}\mathbf{s}\,\Gamma(\mathrm{d}\mathbf{s}),\quad \alpha\neq 1.$$

Now consider inverting the characteristic function $\Phi_{\alpha,d}(\mathbf{t})$ to evaluate the multivariate stable density. The useful idea of Nolan [12] and Abdul-Hamid and Nolan [1] is to perform this inversion with respect to the polar coordinates. The projection of an α-stable random vector onto one-dimensional stable random variable is explained as follows. Here we follow the arguments of Ex.2.3.4 of Samorodnitsky and Taqqu [15]. For any $\mathbf{t}\in\mathbb{R}^d$ consider the characteristic function $E\exp(\mathrm{i}u\langle\mathbf{t},\mathbf{x}\rangle)$ of a linear combination $Y=\sum_i t_iX_i$, i.e.,