Journal of Multivariate Analysis 100 (2009) 2195–2213

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Convex and star-shaped sets associated with multivariate stable distributions, I: Moments and densities

Ilya Molchanov

Department of Mathematical Statistics and Actuarial Science, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland

ARTICLE INFO

Article history:
Received 22 March 2008
Available online 15 April 2009

AMS subject classifications:
60E07
60D05
52A21

Keywords:
Convex body
Generalised function
Fourier transform
Multivariate stable distribution
Star body
Spectral measure
Support function
Zonoid

ABSTRACT

It is known that each symmetric stable distribution in $\mathbb{R}^d$ is related to a norm on $\mathbb{R}^d$ that makes $\mathbb{R}^d$ embeddable in $L_p([0, 1])$. In the case of a multivariate Cauchy distribution the unit ball in this norm is the polar set to a convex set in $\mathbb{R}^d$ called a zonoid. This work interprets symmetric stable laws using convex or star-shaped sets and exploits recent advances in convex geometry in order to come up with new probabilistic results for multivariate symmetric stable distributions. In particular, it provides expressions for moments of the Euclidean norm of a stable vector, mixed moments and various integrals of the density function. It is shown how to use geometric inequalities in order to bound important parameters of stable laws. Furthermore, covariation, regression and orthogonality concepts for stable laws acquire geometric interpretations.

1. Introduction

Since P. Lévy it is well known that the characteristic function of a symmetric stable law in $\mathbb{R}^d$ can be represented as the exponential of the norm as $\varphi(u) = e^{-\|u\|^\alpha}$, where α is the characteristic exponent of the stable law. It is also well known (see [1,2] and [3, Lemma 6.4]) that all norms which might appear in this representation make $(\mathbb{R}^d, \|\cdot\|)$ isometrically embeddable in the space $L_p([0, 1])$ with $p = \alpha \in (0, 2]$.

Each *norm* in $\mathbb{R}^d$ gives rise to the corresponding *unit ball* F. In this paper we neglect the convexity property of the norm and use the term norm, where other works sometimes use the term *gauge function*. If the norm is convex, it can be realised as the support function $\|u\| = h(K, u)$ for the set K polar to the unit ball F, see Section 2. It is known from convex geometry (see [4] for the standard reference) that unit balls in spaces embeddable in $L_1([0, 1])$ are exactly polar sets to *zonoids*. Recall that zonoids appear as Hausdorff limits of zonotopes, i.e. finite Minkowski sums of segments. In application to stable laws with characteristic exponent $\alpha = 1$, we have that $\varphi(u) = e^{-h(K,u)}$ is a characteristic function (necessarily corresponding to the multivariate Cauchy distribution) if and only if K is a zonoid. It is known that all planar centrally symmetric convex sets are zonoids, while this is no longer the case in dimensions 3 and more. This corresponds to a result of Ferguson [5], who showed that the dependency structure of symmetric *bivariate* Cauchy distributions can be described using *any* norm in $\mathbb{R}^2$, while such a representation is no longer possible for *all* norms in dimensions three and more.

It is explained in Section 3 that the correspondence between norms and stable laws can be extended to include all symmetric stable laws by representing their characteristic functions using (possibly non-convex) norms. The unit ball F

E-mail address: ilya@stat.unibe.ch.

0047-259X/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmva.2009.04.003

in the corresponding norm is a star-shaped set called the star body associated with the stable law. Thus, each symmetric stable distribution is uniquely determined by a star body F and the value of the characteristic exponent α and the paper aims to show that this geometric interpretation is useful in the studies of multivariate stable laws. In particular, F is an ellipsoid if and only if the underlying distribution is sub-Gaussian. Section 4 shows that if the characteristic exponent α is at least one, then the norm is convex and the support function representation is also possible using the associated zonoid K being the polar set to F. This associated zonoid K is called L_p-zonoid (where $p = \alpha$), since K can be represented as the Hausdorff limit of a power sum of segments, i.e. the support function of K is the p-mean of the support functions of the summands. Part II of this work deals with one-sided stable laws, including max-stable laws studied in [6].

While the general correspondence between zonoids and stable laws is well understood, this paper concentrates on further relationships between probabilistic aspects of stable laws and geometric properties of associated star-shaped and convex sets. The core of the paper begins in Section 5, where it is shown how to relate the value of the probability density function f of the symmetric stable law at zero to the volume of the associated star body F. It is shown how derivatives of f at the origin are related to further geometric properties of F, in particular to certain ellipsoids associated with F. It also provides an expression for the Rényi entropy of symmetric stable laws. Using geometric results on approximation of convex sets with ellipsoids, Section 5 ends up with a result that gives an estimate for the quality of approximation of a symmetric stable law with a sub-Gaussian one.

Section 6 uses the Fourier analysis for generalised functions together with the geometric representation of the characteristic function in order to compute a number of important characteristics of symmetric stable laws: the moments of the norm of a stable random vector, which previously were known only in the isotropic case, mixed moments of (possibly signed) powers of the coordinates, integrals of the density over subspaces, etc. This section also presents a number of inequalities for the moments and settles the equality cases. Finally, it clarifies a relationship between zonoids of stable laws and zonoids of random vectors studied in [7] and mentions relevant statistical applications.

Section 7 describes several operations with associated star bodies and zonoids and their probabilistic meaning. Using recent approximation results from convex geometry, it is proved that each symmetric stable law can be obtained as the limit for sums of sub-Gaussian laws. It also mentions optimisation ideas, which appear, e.g. in optimising a portfolio whose components have jointly stable distribution.

A geometric interpretation of the covariation is given in Section 8. It also discusses the regression problem for symmetric stable laws, in particularly, the linearity property of multiple regression, which goes back to W. Blaschke's characterisation theorem for ellipsoids. Section 9 discusses the concept of James orthogonality for symmetric stable random variables, which is also extended to define orthogonality of symmetric stable random vectors.

This work attempts to highlight novel relationships between convex geometry and the theory of stable distributions. Further developments are surely possible by invoking other recent results on isotropic bodies, geometric concentration inequalities, or properties of convex sets in spaces of high (but finite) dimension.

2. Star bodies and convex sets

A set F in $\mathbb{R}^d$ is *star-shaped* if $[0, u] \subset F$ for each $u \in F$. A closed bounded set F is called a *star body* if for every $u \in F$ the interval $[0, u)$ is contained in the interior of F and the *Minkowski functional* (or the gauge function) of F defined by

$$\|u\|_F = \inf\{s \geq 0 : u \in sF\}$$

is a continuous function of $u \in \mathbb{R}^d$. The set F can be recovered from its Minkowski functional by

$$F = \{u : \|u\|_F \leq 1\},$$

while the *radial function*

$$\rho_F(u) = \|u\|_F^{-1}$$

provides the polar coordinate representation of the boundary of F for u from the unit *Euclidean sphere* $\mathbb{S}^{d-1}$. In the following we usually consider origin-symmetric star-shaped sets and call them *centred* in this case. If the star body F is centred and *convex*, then $\|u\|_F$ becomes a convex norm on $\mathbb{R}^d$.

The ℓ_p*-ball* in $\mathbb{R}^d$ is defined by

$$B_p^d = \{x \in \mathbb{R}^d : \|x\|_p \leq 1\},$$

where $\|x\|_p = (|x_1|^p + \cdots + |x_d|^p)^{1/p}$ for $p \neq 0$, i.e. $\|x\|_p = \|x\|_F$ with $F = B_p^d$.

A convex set K in $\mathbb{R}^d$ is called a *convex body* if K is compact and has non-empty interior. We usually use the letter F for star bodies (which are not necessarily convex) and K for convex bodies.

The *support function* of a bounded set A in $\mathbb{R}^d$ is defined by

$$h(A, u) = \sup\{\langle x, u\rangle : x \in A\}, \quad u \in \mathbb{R}^d. \tag{2.1}$$

Clearly, $h(A, u)$ coincides with the support function of the convex hull of A.