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Likelihood ratio order of the second order statistic from independent heterogeneous exponential random variables[☆]

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ABSTRACT

Let $X_1, \ldots, X_n$ be independent exponential random variables with respective hazard rates $\lambda_1, \ldots, \lambda_n$, and let $Y_1, \ldots, Y_n$ be independent exponential random variables with common hazard rate λ. This paper proves that $X_{2:n}$, the second order statistic of $X_1, \ldots, X_n$, is larger than $Y_{2:n}$, the second order statistic of $Y_1, \ldots, Y_n$, in terms of the likelihood ratio order if and only if

$$\lambda \geq \frac{1}{2n-1}\left(2\Lambda_1 + \frac{\Lambda_3 - \Lambda_1\Lambda_2}{\Lambda_1^2 - \Lambda_2}\right)$$

with $\Lambda_k = \sum_{i=1}^n \lambda_i^k$, $k = 1, 2, 3$. Also, it is shown that $X_{2:n}$ is smaller than $Y_{2:n}$ in terms of the likelihood ratio order if and only if

$$\lambda \leq \frac{\sum_{i=1}^n \lambda_i - \max_{1\leq i\leq n} \lambda_i}{n-1}.$$

These results form nice extensions of those on the hazard rate order in Păltănea [E. Păltănea, On the comparison in hazard rate ordering of fail-safe systems, Journal of Statistical Planning and Inference 138 (2008) 1993–1997].

1. Introduction

Order statistics play an important role in statistics, reliability theory, and many applied areas. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistics of random variables $X_1, X_2, \ldots, X_n$. It is well known that the kth order statistic $X_{k:n}$ is the lifetime of an $(n-k+1)$-out-of-n system, which is a very popular structure of redundancy in fault-tolerant systems and has been applied in industrial and military systems. In particular, the lifetimes of parallel and series systems correspond to the order statistics, $X_{n:n}$ and $X_{1:n}$. Order statistics have been extensively investigated in the case when the observations are independent and identically distributed (i.i.d). However, in some practical situations, observations are non-i.i.d. Due to the complicated expression of the distribution in the non-i.i.d case, only limited results are found in the literature. One may refer to [1–3] for comprehensive discussions on this topic, and the recent review article of [4] for results on the independent and non-identically distributed (i.ni.d) case.

Due to the nice mathematical form and the unique memoryless property, the exponential distribution has widely been applied in statistics, reliability, operations research, life testing and other areas. Readers may refer to [5,6] for an encyclopedic

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treatment to developments on the exponential distribution. Here, we will focus on the second order statistic, which characterizes the lifetime of the $(n-1)$-out-of-n system (referred as the fail-safe system) in reliability theory and yields the winner's price for the bid in the second price reverse auction; see, for example, [7,8]. Pledger and Proschan [9] were among the first to compare stochastically the order statistics of non-i.i.d exponential random variables with corresponding order statistics from i.i.d exponential random variables. Since then, many researchers followed up this topic including [10–18]. It is also worth noting that some stochastic comparison results on order statistics from two i.ni.d. (not necessarily exponential) samples are closely related to the subject matter of this paper. For instance, [19] proved that the order statistics from two different collections of random variables are ordered in the likelihood ratio order if the underlying random variables are so ordered. Similar results were also given by [20].

For ease of reference, let us first recall some stochastic orders which are closely related to the main results to be developed here in this paper. Throughout, the term *increasing* stands for *monotone non-decreasing* and the term *decreasing* stands for *monotone non-increasing*.

Definition 1.1. For two random variables X and Y with their densities f, g and distribution functions F, G, let $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$. As the ratios in the statements below are well defined, X is said to be smaller than Y in the:

(i) likelihood ratio order (denoted by $X \le_{lr} Y$) if $g(x)/f(x)$ is increasing in x;

(ii) hazard rate order (denoted by $X \le_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x;

(iii) stochastic order (denoted by $X \le_{st} Y$) if $\bar{G}(x) \ge \bar{F}(x)$.

For a comprehensive discussion on stochastic orders, one may refer to [21,22].

Suppose $X_1, \ldots, X_n$ are independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from an exponential distribution with common hazard rate λ. Bon and Păltănea [16] showed, for $1 \le k \le n$,

$$X_{k:n} \ge_{st} Y_{k:n} \iff \lambda \ge \left[\binom{n}{k}^{-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \ldots \lambda_{i_k} \right]^{\frac{1}{k}}. \tag{1.1}$$

Recently, [18] further proved

$$X_{2:n} \ge_{hr} Y_{2:n} \iff \lambda \ge \check{\lambda} = \sqrt{\binom{n}{2}^{-1} \sum_{1 \le i < j \le n} \lambda_i \lambda_j} \tag{1.2}$$

and

$$X_{2:n} \le_{hr} Y_{2:n} \iff \lambda \le \frac{\sum_{i=1}^{n} \lambda_i - \max_{1 \le i \le n} \lambda_i}{n-1}. \tag{1.3}$$

These two results partially improve (1.1) in the case with $k = 2$.

This paper investigates the likelihood ratio order instead. In this regard, it is proved that

$$X_{2:n} \ge_{lr} Y_{2:n} \iff \lambda \ge \hat{\lambda} = \frac{1}{2n-1} \left(2\Lambda_1 + \frac{\Lambda_3 - \Lambda_1 \Lambda_2}{\Lambda_1^2 - \Lambda_2} \right) \tag{1.4}$$

with $\Lambda_k = \sum_{i=1}^{n} \lambda_i^k$, $k = 1, 2, 3$. Also,

$$X_{2:n} \le_{lr} Y_{2:n} \iff \lambda \le \frac{\sum_{i=1}^{n} \lambda_i - \max_{1 \le i \le n} \lambda_i}{n-1}. \tag{1.5}$$

These two results form nice extensions of (1.2) and (1.3). Some examples of special cases are given for illustrating the performance of our main results.

2. Preliminaries

This section presents some useful lemmas, which are not only helpful for proving our main results in what follows, but are also of independent interest.

Definition 2.1. Suppose two vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. Let $x_{(1)} \le \cdots \le x_{(n)}$ and $y_{(1)} \le \cdots \le y_{(n)}$ be the increasing arrangements of their components, respectively.