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Bivariate generalized exponential distribution

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ABSTRACT

Recently it has been observed that the generalized exponential distribution can be used quite effectively to analyze lifetime data in one dimension. The main aim of this paper is to define a bivariate generalized exponential distribution so that the marginals have generalized exponential distributions. It is observed that the joint probability density function, the joint cumulative distribution function and the joint survival distribution function can be expressed in compact forms. Several properties of this distribution have been discussed. We suggest to use the EM algorithm to compute the maximum likelihood estimators of the unknown parameters and also obtain the observed and expected Fisher information matrices. One data set has been re-analyzed and it is observed that the bivariate generalized exponential distribution provides a better fit than the bivariate exponential distribution.

1. Introduction

Gupta and Kundu [5] introduced the generalized exponential (GE) distribution as a possible alternative to the well known gamma or Weibull distribution. The generalized exponential distribution has lots of interesting properties and it can be used quite effectively to analyze several skewed life time data. In many cases it is observed that it provides better fit than Weibull or gamma distributions. Since the distribution function of the GE is in closed form, it can be used quite easily for analyzing censored data also. The frequentest and Bayesian inferences have been developed for the unknown parameters of the GE distribution. The readers are referred to the review article of Gupta and Kundu [6] for a current account on GE distribution.

Although quite a bit of work has been carried out in the recent years on GE distribution, not much attempt has been made to extend this to the multivariate set up. Recently Sarhan and Balakrishnan [10] has defined a new bivariate distribution using the GE distribution and exponential distribution and derived several interesting properties of this new distribution. Although they obtained the new bivariate distribution from the GE and exponential distributions, but the marginal distributions are not in known forms. In fact it is not known to the authors the existence of any bivariate distribution whose marginals are generalized exponential distributions.

The main aim of this paper is to provide a bivariate generalized exponential (BVGE) distribution so that the marginal distributions are GE distributions. In this connection, it may be mentioned here that Arnold [2] provided some general techniques to construct multivarite distribution with specified marginals. We have adopted one of those techniques. The proposed BVGE distribution has three parameters but the scale and location parameters can be easily introduced. The joint cumulative distribution function (CDF), the joint probability density function (PDF) and the joint survival distribution function (SDF) are in closed forms, which make it convenient to use in practice.

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The maximum likelihood estimators (MLEs) can be used to estimate the four unknown parameters when the scale parameter is also present. Although, the MLEs as expected can not be obtained in explicit forms, but the EM algorithm can be used quite effectively to obtain the MLEs. We also provide the observed and expected Fisher information matrices for practical users. Recently Meintanis [9] analyzed one data and concluded that bivariate [8] exponential distribution provided a very good fit. We have re-analyzed the same data set and have observed that the proposed BVGE distribution provides a much better fit than the Marshal and Olkin bivariate exponential model and also have provided some justification.

This article is organized as follows. In Section 2, we define the BVGE distribution and discuss its different properties. The EM algorithm to compute the MLEs of the unknown parameters is provided in Section 3. The analysis of a data set is provided in Section 4. Finally, we conclude the paper in Section 5.

2. Bivariate generalized exponential distribution

The univariate GE distribution has the following CDF and PDF respectively for $x > 0$;

$$F_{GE}(x;\alpha,\lambda) = \left(1-e^{-\lambda x}\right)^{\alpha}, \quad f_{GE}(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}. \tag{1}$$

Here $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters respectively. It is clear that for $\alpha = 1$, it coincides with the exponential distribution. From now on a GE distribution with the shape parameter α and the scale parameter λ will be denoted by $\text{GE}(\alpha, \lambda)$. For brevity when $\lambda = 1$, we will denote it by $\text{GE}(\alpha)$ and for $\alpha = 1$, it will be denoted by $\text{Exp}(\lambda)$.

From now on unless otherwise mentioned, it is assumed that $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0, \lambda > 0$. Suppose $U_1 \sim \text{GE}(\alpha_1, \lambda)$, $U_2 \sim \text{GE}(\alpha_2, \lambda)$ and $U_3 \sim \text{GE}(\alpha_3, \lambda)$ and they are mutually independent. Here '$\sim$' means follows or has the distribution. Now define $X_1 = \max\{U_1, U_3\}$ and $X_2 = \max\{U_2, U_3\}$. Then we say that the bivariate vector (X_1, X_2) has a bivariate generalized exponential distribution with the shape parameters α_1, α_2 and α_3 and the scale parameter λ. We will denote it by $\text{BVGE}(\alpha_1, \alpha_2, \alpha_3, \lambda)$. Now for the rest of the discussions for brevity, we assume that $\lambda = 1$, although the results are true for general λ also. The BVGE distribution with $\lambda = 1$ will be denoted by $\text{BVGE}(\alpha_1, \alpha_2, \alpha_3)$. Before providing the joint CDF or PDF, we first mention how it may occur in practice.

Stress model

Suppose a system has two components. Each component is subject to individual independent stress say U_1 and U_2 respectively. The system has an overall stress U_3 which has been transmitted to both the components equally, independent of their individual stresses. Therefore, the observed stress at the two components are $X_1 = \max\{U_1, U_3\}$ and $X_2 = \max\{U_2, U_3\}$ respectively.

Maintenance model

Suppose a system has two components and it is assumed that each component has been maintained independently and also there is an overall maintenance. Due to component maintenance, suppose the lifetime of the individual component is increased by U_i amount and because of the overall maintenance, the lifetime of each component is increased by U_3 amount. Therefore, the increased lifetimes of the two component are $X_1 = \max\{U_1, U_3\}$ and $X_2 = \max\{U_2, U_3\}$ respectively.

The following results will provide the joint CDF, joint PDF and conditional PDF.

Theorem 2.1. *If* $(X_1, X_2) \sim \text{BVGE}(\alpha_1, \alpha_2, \alpha_3)$, *then the joint CDF of* (X_1, X_2) *for* $x_1 > 0, x_2 > 0$, *is*

$$F_{X_1,X_2}(x_1, x_2) = (1-e^{-x_1})^{\alpha_1}(1-e^{-x_2})^{\alpha_2}(1-e^{-z})^{\alpha_3}, \tag{2}$$

where $z = \min\{x_1, x_2\}$.

Proof. Trivial and therefore it is omitted. ■

Corollary 2.1. *The joint CDF of the* $\text{BVGE}(\alpha_1, \alpha_2, \alpha_3)$ *can also be written as*

$$\begin{aligned} F_{X_1,X_2}(x_1, x_2) &= F_{GE}(x_1, \alpha_1)F_{GE}(x_2, \alpha_2)F_{GE}(z; \alpha_3) \\ &= \begin{cases} F_{GE}(x_1, \alpha_1+\alpha_3)F_{GE}(x_2, \alpha_2) & \text{if} \quad x_1 < x_2 \\ F_{GE}(x_1, \alpha_1)F_{GE}(x_2, \alpha_2+\alpha_3) & \text{if} \quad x_2 < x_1 \\ F_{GE}(x, \alpha_1+\alpha_2+\alpha_3) & \text{if} \quad x_1 = x_2 = x. \end{cases} \end{aligned}$$

Theorem 2.2. *If* $(X_1, X_2) \sim \text{BVGE}(\alpha_1, \alpha_2, \alpha_3)$, *then the joint PDF of* (X_1, X_2) *for* $x_1 > 0, x_2 > 0$, *is*

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & \text{if } 0 < x_1 < x_2 < \infty \\ f_2(x_1, x_2) & \text{if } 0 < x_2 < x_1 < \infty \\ f_0(x) & \text{if } 0 < x_1 = x_2 = x < \infty, \end{cases}$$