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On the conic section fitting problem

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Abstract

Adjusted least squares (ALS) estimators for the conic section problem are considered. Consistency of the translation invariant version of ALS estimator is proved. The similarity invariance of the ALS estimator with estimated noise variance is shown. The conditions for consistency of the ALS estimator are relaxed compared with the ones of the paper Kukush et al. [Consistent estimation in an implicit quadratic measurement error model, *Comput. Statist. Data Anal.* 47(1) (2004) 123–147].

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1. Introduction

The problem considered in this paper is to estimate a hypersurface of the second order that fits a sample of points $x_1, x_2, \ldots, x_m$ in $\mathbb{R}^n$. A second order surface in $\mathbb{R}^n$ is described by the equation

$$x^\top Ax + b^\top x + d = 0. \tag{1}$$

Without loss of generality, one can assume the matrix A to be symmetric. Let $\mathbb{S}$ be a set of real $n \times n$ symmetric matrices. The set of all the triples (A, b, d) is $\mathbb{V} := \mathbb{S} \times \mathbb{R}^n \times \mathbb{R}$.

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Consider a measurement error model. Assume that $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m$ lie on the true surface $\{x \mid x^\top \bar{A} x + \bar{b}^\top x + \bar{d} = 0\}$. In Section 4 we consider the structural case where $\bar{x}_1, \dots, \bar{x}_m$ is an independent identically distributed sequence, while in the rest of the paper the model is functional, i.e., $\bar{x}_1, \dots, \bar{x}_m$ are nonrandom. The true values are observed with errors, which give the measurements $x_1, \dots, x_m$. The measurement errors are supposed to be identically distributed normal variables, the variance of which is either specified or unknown. The parameters of the true surface are parameters of interest. The conic section estimation problem arises in computer vision and meteorology, see [4] and [5].

We use the word "conic" in a very wide sense. Any set that can be defined by Eq. (1) is referred to as "conic". "The true conic" is neither the entire space $\mathbb{R}^n$ nor a subset of a hyperplane, and our conditions ensure that.

Consider the ordinary least squares (OLS) estimator, which is defined by the minimization of the loss function

$$Q_{\text{ols}}(A, b, d) := \sum_{l=1}^{m} (x_l^\top A x_l + b^\top x_l + d)^2.$$

It is easy to compute but inconsistent in the errors-in-variables setup.

The orthogonal regression estimator is inconsistent as well, though it has smaller asymptotic bias [2, Example 3.2.4].

To reduce the asymptotic bias, the renormalization procedure can be used, see [4]. In [5] an adjusted loss function $Q_{\text{als}}(\beta)$ is defined implicitly via the equation

$$\mathbb{E} Q_{\text{als}}(A, b, d) = \sum_{l=1}^{m} (\bar{x}_l^\top A \bar{x}_l + b^\top \bar{x}_l + d)^2,$$

and consistency of the resulting ALS estimator is proved. A computational algorithm and a simulation study for the method of [5] are given in [7].

The ALS estimator with known error variance is not translation-invariant. In this paper we propose a translation-invariant modification of the ALS estimator (TALS estimator). Its consistency is shown. The translation invariance of the ALS estimator with estimated error variance is proved as well, and the conditions for consistency of the estimator are relaxed.

We propose a definition of invariance of an estimator. By appropriate choice of the parameter space and the estimation space this definition can be deduced from the definition of equivariance given in [6, Section 3.2].

The Euclidean norm of a vector $x = (x_1, \dots, x_d)^\top \in \mathbb{R}^d$ is denoted by $\|x\| := \sqrt{\sum_{i=1}^d x_i^2}$. If $A = (a_{i,j})$ is $m \times n$ matrix, $\|A\| := \max_{\|x\| \leqslant 1} \|Ax\|$, while $\|A\|_F := \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}$ is its Frobenius norm. The rank of the matrix A is denoted by $\operatorname{rk} A$. If $m = n$, then $\operatorname{tr} A := \sum_{i=1}^m a_{ii}$ is the trace of A.

As a direct sum of three Euclidean spaces, $\mathbb{V}$ is a Euclidean space with inner product

$$\langle (A_1, b_1, d_1),\ (A_2, b_2, d_2) \rangle := \operatorname{tr}(A_1 A_2) + b_1^\top b_2 + d_1 d_2.$$

The induced norm is

$$\|(A, b, d)\| := \sqrt{\|A\|_F^2 + \|b\|^2 + d^2}.$$