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# Subsampling for continuous-time almost periodically correlated processes

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ABSTRACT

In this work we investigate the problem of consistency of subsampling procedure for estimators in continuous time nonstationary stochastic processes with periodic or almost periodic covariance structure. The motivation for this work comes from the difficulty associated with handling the asymptotic distributions corresponding to estimates of second order characteristics for such nonstationary processes. It is shown that an appropriately normalized estimator has a consistent subsampling version provided that some mild regularity conditions are fulfilled. We also prove the mean square and almost everywhere consistency of our subsampling procedure. As a result of the research, we are able to construct the subsampling-based confidence intervals for the relevant characteristics of such nonstationary processes. We show that our results can be generalized to other nonstationary continuous time processes. At the end of the paper, simulations and real data applications are considered.



## 1. Introduction

The main objective of this paper is to provide a construction of subsampling procedure that works well for continuous-time harmonizable nonstationary stochastic processes with almost periodic second-order characteristics (HAPC processes). In recent research on nonstationary stochastic processes and their applications, the class of HAPC processes has received considerable attention (see Antoni, 2009; Hurd, 1991; Lii and Rosenblatt, 2002, 2006).

In a survey paper on the topic, Gardner et al. (2006) have classified majority of recently published research papers that deal with probabilistic, statistical and applied aspects of such nonstationary phenomena. It is now well known that such models are widely applicable in communication signal processing, vibromechanics, econometrics, climatology and biology. Statistical inference in such models has one fundamental difficulty related to distributions of estimation procedures. If one wants to estimate the second order characteristics of HAPC processes, the estimators will have asymptotic Gaussian distribution which variance is unknown and difficult to compute (see e.g. Dehay and Leśkow, 1996). Fortunately, recent research on resampling procedures for stationary continuous time stochastic processes has suggested methods that can be possibly introduced to HAPC processes. What one wants is to construct confidence intervals for unknown moment characteristics without invoking asymptotic distributions. A number of results are already available for nonstationary

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discrete-time processes, including periodically correlated time series (see Lenart et al., 2008). It is not yet very well known, though, how to introduce a concept of resampling into *continuous-time HAPC processes*. One reason is the essential difficulty in reconstructing a sample in such case. Another is difficulty in proving consistency of the introduced method. In this paper we propose a subsampling method for HAPC processes and we prove the consistency of this method under some mild assumptions on the underlying process. Our method can be also applied to more general classes of nonstationary processes. One of such examples is the class of *generalized almost periodically correlated* (GAPC) *processes* . This class has been recently actively studied in signal processing (see Izzo and Napolitano, 1998, 2005). To motivate our study, we will also provide examples based on simulations and real data coming from the study of human locomotion, more specifically a study of the ground-reaction force (GRF) signal – see Sabri et al. (2010) for details.

Informally, consistency of subsampling means that this method generates valid quantiles for confidence intervals and hypothesis tests in nonstationary models under very mild regularity conditions. Once consistency is established, one can compute the confidence intervals and critical values from the subsampling distributions instead of the asymptotic distributions. Therefore, there is a strong motivation to have a closer look at finite-sample distributions of the estimators. This paper provides the fundamental result on that for HAPC processes and poses open questions leading to further research.

*The synopsis of the paper*: In Section 2, we give some background results. In particular, we recall the notion of harmonizable almost periodically correlated processes (HAPC processes) and give motivating examples to study such models. Moreover, we present the technique of subsampling (see Politis and Romano, 1994; Politis et al., 1999) to the case of HAPC processes. Section 3 is devoted to showing the consistency of the introduced subsampling method for HAPC processes. We show the subsampling consistency that is almost everywhere and quadratic mean convergence. We also present an extension of our results to the class of generalized almost periodically correlated processes. In Section 4 we discuss an application of our results to the construction of the confidence intervals for the cyclic covariance and we show the application to human locomotion signal GRF. The Appendix containing proofs of our results is located at the end of our paper.

## 2. Background

### 2.1. Harmonizable almost periodically correlated (HAPC) processes

For the convenience of the reader we recall some basic facts from the theory of nonstationary harmonizable almost periodically correlated (HAPC) processes (see Hurd, 1991; Lii and Rosenblatt, 2002, 2006). With no loss of generality, we assume that the underlying stochastic process $X=\{X(t): t\in\mathbb{R}\}$ is zero mean.

**Definition 1.** A zero mean, real valued process $X=\{X(t): t\in\mathbb{R}\}$ observed on $(\Omega,\mathcal{F},P)$ is defined as *harmonizable almost periodically correlated* (HAPC) process, when:

(i) The shifted covariance $B(t,\tau)=cov\{X(t),X(t+\tau)\}=E\{X(t)X(t+\tau)\}$ is almost periodic in $t$ for any $\tau$ in $\mathbb{R}$.
(ii) There exists a spectral measure $M$ on $\mathbb{R}^2$ such that

$$B(t,\tau)=\int\int_{\mathbb{R}^2}\exp(i\alpha t)\exp(-i\alpha'(t+\tau))\,dM(\alpha,\alpha').$$

Harmonizability is the existence of the spectral measure $M$ representing the covariance $B(t,\tau)$. It provides a natural tool for the frequency domain analysis of nonstationary processes introducing the concept of *bispectrum* (see also Antoni, 2009). Our approach is in the time domain and we assume harmonizability for the sake of simplicity and clarity of presentation. The results of the paper can be easily generalized to stochastic processes fulfilling (i) but not necessarily fulfilling (ii) (see e.g. Gladyshev, 1963).

From (i) we have that there exists a sequence of trigonometric polynomials which converges uniformly in $t\in\mathbb{R}$ to $B(t,\tau)$ (see Dehay, 1994; Gladyshev, 1963; Hurd, 1991).

The spectral measure $M$ of the HAPC process $\{X(t): t\in\mathbb{R}\}$ is concentrated on the union of straight lines $D_\lambda=\{(\alpha,\alpha')\in\mathbb{R}^2 : \alpha-\alpha'=\lambda\}$, $\lambda\in\Lambda$, where $\Lambda=\{\lambda : a(\lambda,\tau)\neq 0\}$ is a finite or countable subset of $\mathbb{R}$ (see Hurd, 1991). Moreover, the shifted covariance function $B(\cdot,\cdot)$ of the HAPC process admits a Fourier–Bohr decomposition:

$$B(t,\tau)=\sum_{\lambda\in\Lambda}a(\lambda,\tau)e^{i\lambda t}, \tag{1}$$

where

$$a(\lambda,\tau)=\int\int_{D_\lambda}e^{-i\alpha'\tau}\,dM(\alpha,\alpha').$$

The convergence of the sum in (1) is absolute and uniform with respect to $t$ and $\tau$. The *cyclic covariance* $a(\lambda,\tau)$ can be obtained as a limit:

$$a(\lambda,\tau)=\lim_{T\to\infty}\frac{1}{T}\int_t^{T+t}B(s,\tau)e^{-i\lambda s}\,ds. \tag{2}$$