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Asymptotic distribution of the delay time in page's sequential procedure

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ABSTRACT

In this paper the asymptotic distribution of the stopping time in Page's sequential cumulative sum (CUSUM) procedure is presented. Page as well as ordinary cumulative sums are considered as detectors for changes in the mean of observations satisfying a weak invariance principle. The main results on the stopping times derived from these detectors extend a series of results on the asymptotic normality of stopping times of CUSUM-type procedures. In particular the results quantify the superiority of the Page CUSUM procedure to ordinary CUSUM procedures in late change scenarios. The theoretical results are illustrated by a small simulation study, including a comparison of the performance of ordinary and Page CUSUM detectors.

1. Introduction

Monitoring the adequacy of stochastic models is undoubtedly of great importance in many areas of application. A change in the dynamics of the model results in misspecifications and consequently influences conclusions drawn from the model output, e.g., forecasts. Since false conclusions generally imply increasing costs, the speed of detection is crucial for the construction of sequential change-point procedures.

As one of the founding fathers of sequential change-point analysis Page (1954) introduced the Page CUSUM detector motivated by certain problems in quality control. In this context the properties of detectors were explored mainly employing constant thresholds. In the literature, a comparison of the speed of detection of this type of procedures in many cases is based on their *average* detection delay. Even optimality criteria that were introduced for such procedures are referring to the *average* run length (ARL) to false alarm. However, procedures designed with respect to this criterion typically stop with (asymptotic) probability one. For further reading on optimality criteria for Page's CUSUM we refer to the work of Lorden (1971). In addition the monograph of Basseville and Nikiforov (1993) gives an extensive overview of the contributions made since the CUSUM procedure was introduced by Page (1954).

Chu et al. (1996) argue that in many contemporary applications, in particular in an economic context, there are no sampling costs under structural stability. They therefore propose an approach, in which the probability of false alarm is controlled. This approach initiated a multitude of contributions to the field of sequential change-point analysis. E.g., Horváth et al. (2004) propose various sequential tests for the stability of a linear model with independent errors. These tests are based on a stopping rule which

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stops as soon as a CUSUM detector crosses a given boundary function. In a time series regression model, Fremdt (2012) proposed —in a similar fashion—a stopping rule given by the first-passage time of Page's CUSUM over this very boundary function. Based on these procedures we will consider a location model, i.e., we will investigate changes in the mean of certain times series. The latter model is, as a special case, included in the time series regression model of Fremdt (2012). To compare the performance of the CUSUM procedure of Horváth et al. (2004) (which in the sequel will be referred to as *ordinary CUSUM*) and the Page CUSUM, we will investigate the limit distributions of the corresponding stopping times for early as well as for late change scenarios. Until now, only few contributions were made regarding the complete asymptotic distribution of the stopping times which obviously provides more information about the behavior of the stopping times than results on the average run length.

Ordinary CUSUM detectors are defined as the partial sum of, e.g., model residuals from the beginning of the monitoring to the present. These procedures have been studied extensively in the literature, cf., e.g., Horváth et al. (2004, 2007) or Aue et al. (2006b). For the location model results on the asymptotic distribution of the ordinary CUSUM procedure in an early change scenario were given by Aue (2003) and Aue and Horváth (2004). Aue et al. (2009) then also provided an extension to a linear regression model. To prove these results on the asymptotic normality of the ordinary CUSUM detector, strong conditions on the time of change as well as on the magnitude of change were imposed. In this context we also want to mention the work of Hušková and Koubková (2005), who introduced monitoring procedures based on quadratic forms of weighted cumulative sums, and Černíková et al. (2013), who showed the asymptotic normality of the corresponding stopping time under assumptions on the time of change similar to those used in Aue and Horváth (2004) and Aue et al. (2009).

Building on the work of Aue and Horváth (2004), we will derive the asymptotic distribution of the Page as well as the ordinary CUSUM procedure under weaker conditions on the time and magnitude of the change. Hereby, we will show that the Page procedure is more robust to the location and the size of the change than ordinary CUSUM procedures. The corresponding limit distributions for the Page CUSUM are novel in this context and provide a classification of the behavior of the stopping time depending on the interplay of the magnitude and location of the change.

The paper is organized as follows. In Section 2 we will introduce our model setting and required assumptions and formulate our main results. Section 3 contains the results of a small simulation study which compares the ordinary and Page CUSUM in various scenarios. Furthermore, it illustrates particularly how the performance of the procedures depends on the time of change. The proofs of our main results from Section 2 are postponed to the appendix.

2. Asymptotic distribution of the stopping times

Let $X_i,\ i=1,2,\ldots$ follow the location (or "change-in-the-mean") model

$$X_i=\begin{cases}\mu+\varepsilon_i, & i=1,\ldots,m+k^*-1,\\ \mu+\varepsilon_i+\Delta_m, & i=m+k^*,\ m+k^*+1,\ldots,\end{cases} \tag{2.1}$$

where μ and Δ_m are real numbers, $1\le k^*<\infty$ denotes the unknown time of change and the $\{\varepsilon_i\}_{i=1,2,\ldots}$ are zero-mean random variables. In this setting, the number m is the length of a so-called *training period*, in which the model is assumed to be stable. This *noncontamination assumption* (cf. Chu et al., 1996) is used to estimate the model parameters and the asymptotics considered here are (if not stated otherwise) with respect to $m\to\infty$. Furthermore we want to allow for certain dependence structures like autoregressive or GARCH-type dependencies. For this purpose, we assume that the random variables ε_i satisfy the following weak invariance principles:

(A1) $\left|\sum_{i=1}^{m}\varepsilon_i\right|=\mathcal{O}_P(\sqrt{m})$

(A2) There is a sequence of Wiener processes $\{W_m(t):t\ge 0\}_{m\ge 1}$ and a positive constant σ such that

$$\sup_{1/m\le t<\infty}\frac{1}{(mt)^{1/\nu}}\left|\sum_{i=m+1}^{m+mt}\varepsilon_i-\sigma W_m(mt)\right|=\mathcal{O}_P(1)\quad\text{with some }\nu>2.$$

Examples for sequences of random variables satisfying Assumptions (A1) and (A2) are given in Aue and Horváth (2004). Besides i.i.d. sequences these include, e.g., martingale difference sequences and certain stationary mixing sequences. Aue et al. (2006b) showed that the class of augmented GARCH processes, which were introduced by Duan (1997), also satisfies Assumptions (A1) and (A2). This class contains most of the conditionally heteroskedastic time series models applied to describe financial time series. A selection of GARCH models from this class can be found in Aue et al. (2006a).

We are interested in testing the hypothesis

$$H_0:\quad \Delta_m=0$$

against the alternatives

$$H_{A,1}:\quad \Delta_m>0\quad\text{and}\quad H_{A,2}:\quad \Delta_m\neq 0.$$