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Optimal blocked orthogonal arrays

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ABSTRACT

Blocking is an important technique to reduce the noises introduced from uncontrollable variables. Many optimal blocking schemes have been proposed in the literature but there is no single approach that can be applied to various blocked designs. In this paper, we construct a mathematical framework using the count function (or the so-called indicator function) and develop a comprehensive methodology which allows us to select various optimal blocked orthogonal arrays: regular or non-regular designs with qualitative, quantitative or mixed-type factors of two, three, higher or mixed levels. Under this framework, most existing approaches are special cases of our methodology.

1. Introduction

Blocking is an important technique in design of experiments to reduce experimental noises which could be introduced from day-to-day, operator-to-operator, shift-to-shift, or lot-to-lot variations. An appropriate blocked design can efficiently alleviate effects of nuisance variables and make analytical results more reliable. The early publications, including National Bureau of Standards (1957) tables and Connor and Zelen (1959), provided several blocked fractional factorial designs, but these designs were not selected based on theoretical or optimal strategies. Recently, Bisgaard (1994a, 1994b), Sun et al. (1997), Sitter et al. (1997), and Chen and Cheng (1999) have focused on theoretical studies and provided blocked designs according to their proposed optimal schemes. Cheng and Wu (2002) argued that some of these strategies violated the *effect hierarchy principle* and proposed their optimal blocking schemes for two-level and three-level blocked fractional factorial designs. However, they ignored the different properties between qualitative and quantitative factors in three-level designs, which were addressed in Cheng and Ye (2004) that 'for designs with quantitative factors, level permutation of one or more factors in a design matrix could result in different geometric structures, and, thus, different design properties'. Therefore, the designs selected by Cheng and Wu (2002) are optimal for qualitative factors but could be unoptimal for quantitative factors (see Section 4).

All of the above approaches focused on blocked regular designs. Recently, non-regular designs, such as Plackett–Burman designs, have received much attention because they possess several advantages over regular designs. For example, non-regular designs are widely applied to the screening experiments because of economy and flexibility of the run size. For studying their properties, many approaches, such as the count function (or the so-called indicator function) (Fontana et al., 2000; Ye, 2003; Cheng and Ye, 2004) and the *J*-characteristics (Deng and Tang, 1999; Tang and Den, 1999, Tang, 2001), were developed to extend word length patterns and minimum aberration criteria to non-regular designs. Through the

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mathematical framework of the count function proposed by Ye (2003), the optimal blocking strategies were successfully extended to two-level blocked non-regular designs by Cheng et al. (2004).

In practice, various experimental designs are required for different experimental purposes. For example, engineers in manufacture factories may want to include quantitative factors, such as temperature or pressure, in experiments to study their quadratic effects and fit response surfaces. In this case, a three-level or mixed-level blocked non-regular orthogonal array with quantitative factors may be desirable. Previous approaches only focused on specific designs: Sitter et al. (1997), Sun et al. (1997) and Cheng and Wu (2002) focused on two-level regular designs and three-level regular designs with qualitative factors; Cheng et al. (2004) focused on two-level non-regular designs. It is infeasible to apply these approaches to more complicated orthogonal arrays. Motivated by various experimental requirements in practical applications, we propose a more comprehensive methodology which allows experimenters to select various optimal blocked orthogonal arrays: regular or non-regular designs with qualitative, quantitative or mixed-type factors of two, three, higher or mixed levels. It can be shown that under the mathematical framework we proposed, most existing approaches are special cases of our methodology.

The reminder of this paper is organized as follows. Section 2 sets up the mathematical framework for studying blocked orthogonal arrays. In Section 2.1, a more general definition of the count function is introduced. Factor types and words are discussed in detail in Section 2.2. The definitions of aliasing and confounding in the count functions are presented in Section 2.3. In Section 3, we develop the optimal blocking schemes for various orthogonal arrays. Aberration criteria are defined in Section 3.1. Geometric isomorphisms of designs with quantitative factors are introduced in Section 3.2. Two applications are presented in Section 4. The first application demonstrates that the blocked designs selected by Cheng and Wu (2002) could be unoptimal for quantitative factors. In the second application, we apply the proposed methodology to blocking the orthogonal arrays from L18 into two, three and six blocks. Concluding remarks are given in Section 5.

2. Mathematical framework

For studying more complicated blocked orthogonal arrays, we construct a mathematical framework using the count function. Under this framework, the blocking schemes discussed in Cheng and Wu (2002) and Cheng et al. (2004) can be extended to various blocked orthogonal arrays.

2.1. Count function

The count function originated from the indicator function which was first introduced and named by Fontana et al. (2000) for studying two-level fractional factorial designs. The original indicator function indicates whether design points appear in a design. It was further extended by Ye (2003) and Cheng and Ye (2004) for counting the number of appearance of design points in designs and can be applied to three- or higher-level designs. This new version of counting design points was then widely applied for studying the properties of experimental designs and renamed the 'count function' by Lin and Cheng (2012). The count function expresses designs in polynomial forms as described below. Let $\mathcal{D}$ be an orthogonal array $OA(N, s_1 s_2 \cdots s_k)$, which is a full factorial design with N runs and k factors, $X_1, X_2, \ldots, X_k$, where $N = s_1 s_2 \cdots s_k$ and the levels of factor X_i are set at $S_i = \{0, 1, \ldots, s_i - 1\}$. For factor X_i, define orthogonal contrasts $c_0^i(x), c_1^i(x), \ldots, c_{s_i-1}^i(x)$ which satisfy

$$\sum_{x \in \{0,1,\ldots,s_i-1\}} c_u^i(x) c_v^i(x) = \begin{cases} 0 & \text{if } u \neq v, \\ s_i & \text{if } u = v. \end{cases}$$

For instance, $(c_0^i(0), c_0^i(1)) = (1, 1)$ and $(c_1^i(0), c_1^i(1)) = (-1, 1)$ if X_i is a two-level factor; $(c_0^i(0), c_0^i(1), c_0^i(2)) = (1, 1, 1)$, $(c_1^i(0), c_1^i(1), c_1^i(2)) = (-\sqrt{3/2}, 0, \sqrt{3/2})$, and $(c_2^i(0), c_2^i(1), c_2^i(2)) = (1/\sqrt{2}, -\sqrt{2}, 1/\sqrt{2})$ if X_i is a three-level factor. Let $\mathcal{T} = S_1 \times S_2 \times \cdots \times S_k$. Define the polynomial term by

$$C_{\mathbf{t}}(\mathbf{x}) = \prod_{i=1}^{k} c_{t_i}^i(x_i)$$

for a design point $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ on $\mathbf{t} = (t_1, t_2, \ldots, t_k) \in \mathcal{T}$. Let $\mathcal{A}$ be a k-factor factorial design in the design space of $\mathcal{D}$, that is, $\forall \mathbf{x} \in \mathcal{A}$, $\mathbf{x} \in \mathcal{D}$. The count function of $\mathcal{A}$ is then defined by

$$F_{\mathcal{A}}(\mathbf{x}) = \sum_{\mathbf{t} \in \mathcal{T}} b_{\mathbf{t}} C_{\mathbf{t}}(\mathbf{x}),$$

where the coefficient of term $C_{\mathbf{t}}(\mathbf{x})$ is obtained by

$$b_{\mathbf{t}} = \frac{1}{N} \sum_{\mathbf{x} \in \mathcal{A}} C_{\mathbf{t}}(\mathbf{x}).$$

The count function counts the number of appearance of design points in designs. For example, the count function of the 2^{3-1} fractional factorial design with the defining relation $I = X_1 X_2 X_3$ is $F(\mathbf{x}) = .5C_{000}(\mathbf{x}) + .5C_{111}(\mathbf{x})$. For design point $\mathbf{x} = (0, 0, 1)$, $C_{000}(\mathbf{x}) = c_0^1(0) c_0^2(0) c_0^3(1) = 1 \times 1 \times 1 = 1$ and $C_{111}(\mathbf{x}) = c_1^1(0) c_1^2(0) c_1^3(1) = (-1) \times (-1) \times 1 = 1$ result in $F(\mathbf{x}) = 1$ which indicates that $\mathbf{x} = (0, 0, 1)$ appears once in the 2^{3-1} design. For design point $\mathbf{x} = (1, 0, 1)$, $C_{000}(\mathbf{x}) = c_0^1(1) c_0^2(0) c_0^3(1) = 1 \times 1 \times 1 = 1$ and $C_{111}(\mathbf{x}) = c_1^1(1) c_1^2(0) c_1^3(1) = 1 \times (-1) \times 1 = -1$ result in $F(\mathbf{x}) = 0$, which indicates that $\mathbf{x} = (1, 0, 1)$ is not a run in