Journal of Statistical Planning and Inference 143 (2013) 1330–1342

Contents lists available at SciVerse ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Distribution theory of quadratic forms for matrix multivariate elliptical distribution

José A. Díaz-García *

Universidad Autónoma Agraria Antonio Narro, Department of Statistics and Computation, 25315 Buenavista, Saltillo, Coahuila, Mexico

ARTICLE INFO

Article history:
Received 29 October 2012
Received in revised form
20 March 2013
Accepted 22 March 2013
Available online 29 March 2013

Keywords:
Quadratic forms
Spherical functions
Jacobians
Elliptical models
Real, complex, quaternion and octonion random matrices

ABSTRACT

This paper proposes the density and characteristic functions of a general matrix quadratic form $\mathbf{X}^*\mathbf{A}\mathbf{X}$, when $\mathbf{A}=\mathbf{A}^*$ is a positive semidefinite matrix, $\mathbf{X}$ has a matrix multivariate elliptical distribution and $\mathbf{X}^*$ denotes the usual conjugate transpose of $\mathbf{X}$. These results are obtained for real normed division algebras. With particular cases we obtained the density and characteristic functions of matrix quadratic forms for matrix multivariate normal, Pearson type VII, t and Cauchy distributions.

1. Introduction

The distribution of a matrix quadratic form in multivariate normal sample $\mathbf{X}'\mathbf{A}\mathbf{X}$ has been studied by diverse authors using zonal, Laguerre and Hayakawa polynomials with matrix argument, in real and complex cases, see Khatri (1966), Hayakawa (1966) and Shah (1970), among others. It is important to observe that these results have been obtained for positive definite matrix quadratic forms, i.e. when $\mathbf{A}$ is a positive definite matrix. In the present study, this condition has been changed and now it is assumed that $\mathbf{A}$ is a positive semidefinite matrix, then $\mathbf{X}'\mathbf{A}\mathbf{X}$ is termed general matrix quadratic form.

In diverse cases, certain statistics are functions of a quadratic form or special types of it, and play a very important role in classical multivariate statistical analysis.

By replacing the matrix multivariate normal distribution with a matrix multivariate elliptical distribution, in classical multivariate analysis one obtains what is now termed generalised multivariate analysis. By analogy, the distribution of a matrix quadratic form in a matrix multivariate elliptical sample plays an equally important role in generalised multivariate analysis, as it is now interesting to study the distribution of a quadratic form assuming a matrix multivariate elliptical distribution. Some results in this context have been obtained for particular matrix quadratic forms, see Fang and Anderson (1990, Chapter II, pp. 137–200).

* Tel.: +52 844 411 0293; fax: +52 844 411 0211.
E-mail address: jadiaz@uaaan.mx

0378-3758/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.jspi.2013.03.024

In general, many results first described in statistical theory are then found in real case, and the version for complex case is subsequently studied. In terms of certain concepts and results derived from abstract algebra, it is possible to propose a unified means of addressing not only real and complex cases but also quaternion and octonion cases.

This paper obtains the density and characteristic functions of a matrix quadratic form of a matrix multivariate elliptical distribution for real normed division algebras. Furthermore, these results are obtained when the matrix of the quadratic form is not necessarily positive definite, which generalises most of the results presented in the literature in this context.

This paper is structured as follows: Section 2 provides some definitions and notation on real normed division algebras, introducing the corresponding matrix multivariate elliptical distributions. Some results for Jacobians are proposed and two are obtained together with an extension of one of the basic properties of zonal polynomials. This is also valid for Jack polynomials for real normed division algebras, termed spherical functions for symmetric cones, and are also obtained. Section 3 then derives the matrix quadratic form density function, and as corollaries, some results for particular elliptical distributions are obtained. In Section 4, the characteristic function of a matrix quadratic form is obtained and some particular cases are studied.

2. Preliminary results

Let us introduce some notation and useful results.

2.1. Notation and real normed division algebras

A comprehensive discussion of real normed division algebras can be found in Baez (2002). For convenience, we shall introduce some notations, although in general we adhere to standard notations.

There are exactly four normed division algebras over $\mathfrak{R}$: real numbers ($\mathfrak{R}$), complex numbers ($\mathfrak{C}$), quaternions ($\mathfrak{H}$) and octonions ($\mathfrak{O}$), see Baez (2002). We take into account that $\mathfrak{R}$, $\mathfrak{C}$, $\mathfrak{H}$ and $\mathfrak{O}$ are the only normed division algebras; moreover, they are the only alternative division algebras, and all division algebras have a real dimension of 1, 2, 4 or 8, which is denoted by β, see Baez (2002, Theorems 1–3). In other branches of mathematics, the parameter $\alpha = 2/\beta$ is used, see Edelman and Rao (2005).

Let $\mathfrak{L}^{\beta}_{m,n}$ be the set of all $n\times m$ matrices of rank $m\leq n$ over $\mathfrak{F}$ with m distinct positive singular values, where $\mathfrak{F}$ denotes a *real finite-dimensional normed division algebra*. Let $\mathfrak{F}^{n\times m}$ be the set of all $n\times m$ matrices over $\mathfrak{F}$. The dimension of $\mathfrak{F}^{n\times m}$ over $\mathfrak{R}$ is βmn. Let $\mathbf{A}\in\mathfrak{F}^{n\times m}$, then $\mathbf{A}^* = \overline{\mathbf{A}}^T$ denotes the usual conjugate transpose.

The set of matrices $\mathbf{H}_1\in\mathfrak{F}^{n\times m}$ such that $\mathbf{H}_1^*\mathbf{H}_1 = \mathbf{I}_m$ is a manifold denoted $\mathcal{V}^{\beta}_{m,n}$, termed the *Stiefel manifold* ($\mathbf{H}_1$, also known as *semi-orthogonal* ($\beta=1$), *semi-unitary* ($\beta=2$), *semi-symplectic* ($\beta=4$) and *semi-exceptional type* ($\beta=8$) matrices, see Dray and Manogue, 1999). The dimension of $\mathcal{V}^{\beta}_{m,n}$ over $\mathfrak{R}$ is $[\beta mn - m(m-1)\beta/2 - m]$. In particular, $\mathcal{V}^{\beta}_{m,m}$ with a dimension over $\mathfrak{R}$, $[m(m+1)\beta/2 - m]$, is the maximal compact subgroup $\mathfrak{U}^{\beta}(m)$ of $\mathcal{L}^{\beta}_{m,m}$ and consists of all matrices $\mathbf{H}\in\mathfrak{F}^{m\times m}$ such that $\mathbf{H}^*\mathbf{H} = \mathbf{I}_m$. Therefore, $\mathfrak{U}^{\beta}(m)$ is the *real orthogonal group* $\mathcal{O}(m)$ ($\beta=1$), the *unitary group* $\mathcal{U}(m)$ ($\beta=2$), *compact symplectic group* $\mathcal{S}p(m)$ ($\beta=4$) or *exceptional type matrices* $\mathcal{O}o(m)$ ($\beta=8$), for $\mathfrak{F}=\mathfrak{R}$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$, respectively.

We denote by $\mathfrak{S}^{\beta}_m$ the real vector space of all $\mathbf{S}\in\mathfrak{F}^{m\times m}$ such that $\mathbf{S}=\mathbf{S}^*$. Let $\mathfrak{P}^{\beta}_m$ be the *cone of positive definite matrices* $\mathbf{S}\in\mathfrak{F}^{m\times m}$. Thus, $\mathfrak{P}^{\beta}_m$ consist of all matrices $\mathbf{S}=\mathbf{X}^*\mathbf{X}$, with $\mathbf{X}\in\mathfrak{L}^{\beta}_{m,n}$; then $\mathfrak{P}^{\beta}_m$ is an open subset of $\mathfrak{S}^{\beta}_m$. Over $\mathfrak{R}$, $\mathfrak{S}^{\beta}_m$ consist of *symmetric* matrices; over $\mathfrak{C}$, *Hermitian* matrices; over $\mathfrak{H}$, *quaternionic Hermitian* matrices (also termed *self-dual matrices*) and over $\mathfrak{O}$, *octonionic Hermitian* matrices. Generically, the elements of $\mathfrak{S}^{\beta}_m$ shall be termed *Hermitian matrices*, irrespective of the nature of $\mathfrak{F}$. The dimension of $\mathfrak{S}^{\beta}_m$ over $\mathfrak{R}$ is $[m(m-1)\beta+2m]/2$.

Let $\mathfrak{D}^{\beta}_m$ be the *diagonal subgroup* of $\mathcal{L}^{\beta}_{m,m}$ consisting of all $\mathbf{D}\in\mathfrak{F}^{m\times m}$, $\mathbf{D}=\operatorname{diag}(d_1,\dots,d_m)$ and let $\mathfrak{T}U^{+\beta}_m$ be the subgroup of all *upper triangular* matrices $\mathbf{T}\in\mathfrak{F}^{m\times m}$ such that $t_{ij}=0$ for $1<i<j<m$ with $t_{ii}>0$, $i=1,\dots,m$.

For any matrix $\mathbf{X}\in\mathfrak{F}^{n\times m}$, $d\mathbf{X}$ denotes the *matrix of differentials* (dx_{ij}). Finally, we define the *measure* or volume element ($d\mathbf{X}$) when $\mathbf{X}\in\mathfrak{F}^{m\times n}$, $\mathfrak{S}^{\beta}_m$, $\mathfrak{D}^{\beta}_m$ or $\mathcal{V}^{\beta}_{m,n}$, see Dumitriu (2002).

If $\mathbf{X}\in\mathfrak{F}^{n\times m}$ then ($d\mathbf{X}$) (the Lebesgue measure in $\mathfrak{F}^{n\times m}$) denotes the exterior product of the βmn functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^{n}\bigwedge_{j=1}^{m} dx_{ij} \quad \text{where } dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.$$

If $\mathbf{S}\in\mathfrak{S}^{\beta}_m$ (or $\mathbf{S}\in\mathfrak{T}U^{+\beta}_m$) then ($d\mathbf{S}$) (the Lebesgue measure in $\mathfrak{S}^{\beta}_m$ or in $\mathfrak{T}U^{+\beta}_m$) denotes the exterior product of the $m(m-1)\beta/2+m$ functionally independent variables,

$$(d\mathbf{S}) = \bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i<j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}.$$

Note that for the Lebesgue measure ($d\mathbf{S}$) defined thus, it is required that $\mathbf{S}\in\mathfrak{P}^{\beta}_m$, that is, $\mathbf{S}$ must be a non-singular Hermitian matrix (Hermitian positive definite matrix).