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Nonlinear recursive estimation of volatility via estimating functions

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ABSTRACT

For certain volatility models, the conditional *moments* that depend on the parameter are of interest. Following Godambe and Heyde (1987), the combined estimating function method has been used to study inference when the conditional mean and conditional variance are functions of the parameter of interest (See Ghahramani and Thavaneswaran [Combining Estimating Functions for Volatility. Journal of Statistical Planning and Inference, 2009, 139, 1449–1461] for details). However, for application purposes, the resulting estimates are nonlinear functions of the observations and no closed form expressions of the estimates are available. As an alternative, in this paper, a recursive estimation approach based on the combined estimating function is proposed and applied to various classes of time series models, including certain volatility models.

1. Introduction

The analysis of nonlinear time series models in financial econometrics continues to be of interest. Many financial series, such as returns on stocks and foreign exchange rates, exhibit leptokurtosis and time-varying volatility. These two features have been the subject of extensive studies ever since Nicholls and Quinn (1982) and Engle (1982) reported them. The random coefficient autoregressive (RCA) models, the autoregressive conditional heteroscedastic (ARCH) models due to Engle (1982), and the GARCH models of Bollerslev (1986) provide a convenient framework to study time-varying volatility in financial markets.

Various volatility models are such that the first two conditional moments of the observed depend on the parameter of interest. An example is the discrete-time process given by Shiryaev (1995) as

$$y_{t+1} = A\theta_t + z_{t+1},$$

$$\theta_{t+1} = a\theta_t + (1+\theta_t)\eta_{t+1},$$

where $\{z_t$ and $\{\eta_t$ are two independent sequences of independent Gaussian random variables with mean zero and variance σ_z^2 and σ_η^2, respectively. An example of a continuous-time stochastic process where the first two conditional moments depend on the same parameter is the Black–Scholes model observed with noise given by Klebaner (2005) of the form

$$dy_t = \theta_t dt + dW_2(t),$$

$$d\theta_t = (\mu + \tfrac{1}{2}\sigma^2)\theta_t dt + \sigma\theta_t dW_1(t),$$

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where $\{W_1(t)$ and $\{W_2(t)$ are two independent Wiener processes. By using the transformation of the form $b_1 = \sqrt{E(\theta_t)^2}$ and $W_1^*(t) = (\theta_t/b_1)W_1(t)$, the model can be written as

$$dy_t = \theta_t dt + dW_2(t),$$

$$d\theta_t = (\mu + \tfrac{1}{2}\sigma^2)\theta_t dt + \sigma b_1 dW_1^*(t).$$

In Gong and Thavaneswaran (2009), recursive estimates for the two aforementioned models have been studied by using the minimum mean square error (MMSE) criterion. In this paper, our aim is to obtain recursive estimators of the parameter in a wider sense, using estimating function theory.

Estimating function theory is well suited to financial data (see Bera et al., 2006 and Pandher, 2001, for example). The combined estimating function method has been studied by Naik-Nimbalkar and Rajarshi (1995) and also by Thompson and Thavaneswaran (1999) in the filtering context (see also McLeish and Small, 1988; Heyde, 1997). Another application of the combined estimating function method for hypothesis testing, based on ARMA models with GARCH errors is given in Ghahramani and Thavaneswaran (2006). Recently, Ghahramani and Thavaneswaran (2009) have studied GARCH model identification by combining least squares and least absolute deviation estimating functions and the method has been applied to identification problems with several real financial datasets. Combined estimating functions had also been studied in Godambe and Heyde (1987).

In this paper, the combined estimating function method is applied to obtain optimal recursive estimates of the parameter in autoregressive models with GARCH errors and of the parameter in RCA models. Combinations of least squares and quadratic estimating functions, as well as combinations of least squares and LAD estimating functions, are considered. The following example motivates the use of estimating function theory for recursive estimation of the parameter in certain time series models.

Consider an AR(1) process

$$y_t = \phi y_{t-1} + \varepsilon_t, \tag{1.1}$$

where $\{\varepsilon_t$ is an i.i.d. sequence of symmetric stable random variables with characteristic function $c(j) = \exp(-|j|^\alpha)$, and $0 < \alpha \le 2$. Closed form representations of the density exist only when $\alpha = 1$ (ε_t follows a Cauchy distribution) or when $\alpha = 2$ (ε_t follows a Gaussian distribution). Moreover, the second moment is not finite when $0 < \alpha < 2$. Interest centers on estimating the parameter ϕ on the basis of the sample $y_1, \ldots, y_n$. Merkouris (2007) recently has proposed the estimating function approach to estimate the parameter ϕ in (1.1). The optimal transformed martingale estimating function equation turns out to be

$$G_n^\star(j) = \frac{2j\exp(2^\alpha|j|^\alpha)}{\exp(|j|^\alpha)(\exp(2^\alpha|j|^\alpha)-1)}\sum_{t=1}^{n} y_{t-1}\sin(j(\phi y_{t-1} - y_t)) = 0. \tag{1.2}$$

The optimal choice of j in $G_n^\star(j)$ is discussed in Merkouris (2007). Recursive (or online) estimation of a parameter where the estimate of the parameter at time $n+1$ is the estimate of the parameter at time n plus an adjustment is advantageous when there is a large stretch of data and observations become available successively over time. Recursive estimation of the parameter based on nonlinear estimating functions had been studied by Thavaneswaran and Heyde (1999) and in this paper, recursive estimation based on the *combined* estimating function is studied.

In Thavaneswaran and Heyde (1999), Thompson and Thavaneswaran (1999) and Naik-Nimbalkar and Rajarshi (1995), the combined estimating function approach had been used in the Bayesian context and using the Bayesian version of the Cramer–Rao lower bound (see Gill and Levit, 1995), the superiority of the approach had been demonstrated. However, in this paper, the combined estimating function approach is used in the non-Bayesian setting.

This paper is organized as follows. In Section 2, optimal nonlinear recursive parameter estimation for time series models where the conditional mean and conditional variance depend on the same parameter is discussed. The combined estimating function method is used; the method utilizes Godambe's (1985) theorem for optimal estimating functions for discrete-time stochastic parameters. In Section 3, examples of optimal nonlinear recursive estimates for a class of AR processes with GARCH errors and, for a class of RCA models are given. In Section 4, optimal estimation is discussed in the pre-filtered estimation context. Section 5 concludes the paper.

2. Recursive estimation using combined estimating functions

In Section 2.1, the theorem on optimal estimating functions for discrete time stochastic processes due to Godambe (1985) is recalled. In the second subsection, we apply Godambe's theorem and obtain optimal combined estimating functions. In the third subsection, an extension of a theorem due to Thavaneswaran and Heyde (1999) on optimal recursive estimation for discrete time stochastic processes based on estimating functions with applications to nonlinear, non-Gaussian time series models is provided.

2.1. Godambe's theorem

Let $\{y_t : t \in \mathcal{N}$ be a discrete time real-valued stochastic process defined on a probability space $(\Omega, \mathcal{A}, \mathcal{F})$, where the index set $\mathcal{N}$ is the set of all positive integers. Suppose that the observations $\{y_1, \ldots, y_n$ are available and the parameter $\theta \in \Theta$,