Journal of Statistical Planning and Inference 142 (2012) 327–335

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Structural changes and unit roots in non-stationary time series

Boris Brodsky [a,*], Boris Darkhovsky [b]

[a] Central Institute for Mathematics and Economics, Moscow, Russia
[b] Institute for Systems Analysis, Moscow, Russia

ARTICLE INFO

Article history:
Received 29 July 2010
Received in revised form
24 July 2011
Accepted 26 July 2011
Available online 4 August 2011

Keywords:
Structural change
Unit root
Non-parametric method

ABSTRACT

The problem of testing hypotheses of a unit root and a structural change in one-dimensional time series is considered. A non-parametric two-step method for solution of the problem is proposed. The method is based upon the modified Kolmogorov–Smirnov statistic. At the first step of this method the hypothesis of stationarity of an obtained sample is tested against a unified alternative of a statistical non-stationarity of a time series (a unit root or a structural change). At the second step of the proposed method, in case of rejecting the stationarity hypothesis at the first step, the hypothesis of an unknown structural change is tested against the alternative of a unit root. We prove that probabilities of errors (false classification of hypotheses) of the proposed method converge to zero as the sample size tends to infinity.

1. Introduction

Models with unit roots and structural changes (change-points) are often used in econometrics and financial mathematics. To test these hypotheses is one of the most difficult problems in the theory of non-stationary time series. As Perron (2005) remarks in his review of the modern state of this theory, a presence of a structural change in data substantially worsens power of all known tests for unit roots (e.g. the ADF (augmented Dickey and Fuller, 1979, 1981), KPSS (Kwiatkowski–Phillips–Schmidt–Shin, 1992) test), and vice versa, testing for a structural change for models with (possible) unit roots is extremely difficult.

In Perron's opinion, the reason for this situation is that most tests for unit roots and structural changes are statistically 'sensitive' to all of these types of non-stationarity. The real problem is to propose statistical tests which can discriminate between unit roots and structural changes (see Perron, 2005) in non-stationary time series models.

In the econometric context, this problem was actual after the Nelson and Plosser (1982) study of 14 macroeconomic time series of the US economy. The authors demonstrated that the ADF test cannot reject the stochastic trend hypothesis for most of these series. After that Rappoport and Reichlin (1989) and especially Perron (1989, 1994) showed that the mere presence of a structural change in the data precludes effective testing of a stochastic trend hypothesis. For example, if a structural shift in level is present in data, then the MLE estimate of the first autoregressive coefficient is asymptotically biased to 1. Therefore, the ADF test with the modified threshold (MacKinnon, 1996) confirms the hypothesis of a stochastic trend in data.

In subsequent papers by Montanes and Reyes (1999, 2000) it was demonstrated that the Phillips and Perron (1988) test for unit roots has similar problems.

The situation becomes even more complicate if instants of structural changes are unknown. Christiano (1992), Banerjee et al. (1992), Zivot and Andrews (1992), Perron and Vogelsang (1992), and Perron (1997) showed that power of all known

* Corresponding author.
E-mail address: bbrodsky@yandex.ru (B. Brodsky).

0378-3758/$ - see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.jspi.2011.07.021

tests for unit roots drops significantly for samples with structural changes at unknown instants. Some preliminary ideas how to deal with these problems were proposed in Perron and Yabu (2009) and Kim and Perron (2005).

So, we conclude that the problem of testing the unit root and the structural change (change-point) hypotheses is very actual for non-stationary time series.

In this paper, we propose a two-step non-parametric method for testing and discriminating hypotheses of a structural change and a unit root based upon the modified Kolmogorov–Smirnov test. At the first step of the proposed method the hypothesis of stationarity of an obtained time series is tested against the unified alternative of non-stationarity of observations (a structural change and a unit root). At the second step (for an accepted hypothesis of non-stationarity) we test the hypothesis of an unknown structural change against the alternative of a unit root. We prove that the probabilities of errors (erroneous testing and classification of hypotheses) for the proposed method tend to zero exponentially with an increasing sample size.

2. Problem statement and assumptions

2.1. Motivation

For motivation of subsequent assumptions, let us consider the following model of a time series which is often used in practice. Suppose that a one-dimensional time series $\{Y_t\}$ is described by the model $AR(p)$, i.e.

$$Y_t = \alpha_1 Y_{t-1} + \cdots + \alpha_p Y_{t-p} + u_t,$$

where $\{u_t\}$ is the sequence of i.i.d.r.v.'s, $\mathbf{E}u_t = 0$, $\sigma_u^2 = \mathbf{E}u_t^2$, $0 < \sigma_u^2 < \infty$ (here and below we denote by $\mathbf{E}(\mathbf{P})$ the mathematical expectation (probability measure)). Suppose that the characteristic polynomial

$$p(z) = 1 - \alpha_1 z - \cdots - \alpha_p z^p, \quad z \in \mathbb{C}$$

has one unit root $p(1) = 0$ and all other roots of this polynomial lie outside the unit circle. Then $p(z) = p^*(z)(1-z)$ and the polynomial $p^*(z)$ has no roots inside the unit circle.

It follows from here that $1/p^*(z)$ exists for $|z| \le 1$. Then $p(L) = p^*(L)\Delta Y_t = u_t$, where L is the delay operator. Since the polynomial $p^*(L)$ can be inverted, we have the following representation:

$$Y_t = Y_{t-1} + \sum_{j \ge 0} \beta_j u_{t-j}$$

for some coefficients $\{\beta_j\}$.

This means that the considered process Y_t can be described by a random walk model with *dependent* errors. Remark that an analogous representation is valid for the process $\{Y_t\}$ described by the $ARMA(p,q)$ model with one unit root.

2.2. Problem statement

Based on Section 2.1, we conclude that most situations of unit roots and structural changes in one-dimensional time series can be described by the following model:

$$e_i = \rho e_{i-1} + u_i, \quad e_0 \equiv 0, \; i = 1, \ldots, N, \tag{1}$$

where $\{u_i\}$ is a sequence of *dependent* i.d.r.v.'s with zero mean and finite variance.

The problem statement is as follows. Consider model (1) and let the observation sample $\{y_i\}_{i=1}^N$ is given. It is necessary to test the following hypotheses:

- the hypothesis of stationarity $\mathbf{H}_0 : y_i = e_i$, $i = 1, \ldots, N$, $|\rho| \le 1-\delta$, where δ is known;
- the hypothesis of a structural change $\mathbf{H}_1$: $y_i = h\mathbb{I}(i \ge [\theta N]) + e_i$, $i = 1, \ldots, N$, $|\rho| \le 1-\delta$ (here and below $\mathbb{I}(A)$ is the indicator function of a set A, $[a]$ is the integer part of a number a).

In this case θ is an unknown parameter (the relative change-point), $0 < a \le \theta \le 1-a < 1$, where a is known constant, h is unknown parameter (the size of a structural change) such that $0 < b \le |h|$, and b is known constant;

- the unit root hypothesis $\mathbf{H}_2 : \rho = 1$, $y_i = e_i$, $i = 1, \ldots, N$. Under hypothesis $\mathbf{H}_1$ the family (under parameters (θ, h)) of probability measures $\{\mathbf{P}_{\theta,h}\}$ is considered. In this case we consider also the problem of the change-point $\hat{n} \stackrel{\text{def}}{=} [\theta N]$ estimation.

2.3. Assumptions

First, let us introduce and remind necessary notations. Consider the probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two σ-algebras contained in $\mathfrak{F}$. Let $L_p(\mathbb{H})$ be a collection of L_p-integrated random variables X measurable with respect to some