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Reference optimality criterion for planning accelerated life testing

Ancha Xu [a,b,*], Yincai Tang [c]

[a] College of Mathematics and Information Science, Wenzhou University, Zhejiang, 325035, China
[b] School of Natural and Applied Sciences, Northwestern Polytechnical University, Xi'an 710072, China
[c] School of Finance and Statistics, East China Normal University, Shanghai, 200241, China

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ABSTRACT

Most of the current literatures on planning accelerated life testing are based on D-optimality criterion and V-optimality criterion. Such methods minimize the generalized asymptotic variance of the maximum likelihood estimators of the model parameters or that of a quantile lifetime. Similarly, the existing Bayesian planning criterion is usually based on the posterior variance of a quantile lifetime. In this paper, we present a framework for a coherent approach for planning accelerated life testing. Our approach is based on the expectation of Shannon information between prior density function and posterior density function, which is also the spirit for deriving reference prior in Bayesian statistics. Thus, we refer to the criterion as the reference optimality criterion. Then the optimal design is selected via the principle of maximizing the expected Shannon information. Two optimization algorithms, one based on large-sample approximation, and the other based on Monte Carlo simulation, are developed to find the optimal plans. Several examples are investigated for illustration.

1. Introduction

Accelerated life testing (ALT) is commonly utilized to obtain failure time data of the product with high reliability. Compared with standard life-testing methods, ALT could greatly save time and expense. In ALT, specimens are tested at high stress levels to induce early failures, and then failure time data collected from ALT are extrapolated to estimate reliability characteristics (e.g., reliability, mean time to failure (MTTF), or some quantile of the lifetime distribution, etc.) at the normal stress condition. The fundamental theories and methodologies for lifetime analysis have been established over the last three decades. See Meeker and Escobar (1998) for a comprehensive survey. However, the precision of the reliability estimation depends on the design of the ALT plans. Nelson (1990) indicated that optimal testing plans allow maximum possible information to be obtained from the tests, and that optimal testing will save 25%–50% specimens compared with unplanned testing according to the same precision of the estimators in the model.

The goal of the ALT optimal design is to find an optimal experimental plan (specifying the number and magnitude of the accelerated stress levels, and the number of items to be tested at these stress levels) according to some criteria. Various optimization criteria have been used to design ALT plans. The most commonly used one is called V-optimality criterion, which develops plans that minimize the asymptotic variance of the maximum likelihood (ML) estimator of a specified quantile lifetime or a function of the model parameters at normal stress condition. In contrast, for a model with two stress

* Corresponding author at: College of Mathematics and Information Science, Wenzhou University, Zhejiang, 325035, China.
E-mail address: xuancha@wzu.edu.cn (A. Xu).

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factors, Escobar and Meeker (1995) presented a criteria called D-optimality criterion that minimizes the determinant of the variance–covariance matrix of the estimates of the model parameters with an attempt to estimate them well on a whole. The motivation of D-optimality criterion is that the volume of an approximate joint confidence region for the model parameters is inversely proportional to an estimate of the square root of the determinant of the variance–covariance matrix. For the scenario of single stress factor, V-optimality criterion is also used by some authors. See Liu and Qiu (2011), Guan and Tang (2012) and Ye et al. (2013). These two criteria are used for different purposes and will lead to different results. Determining the optimal plans usually needs the initial values of the model parameters based on the two criteria. The initial values of the model parameters could be obtained from previous experience with similar products and materials, or engineering judgment. Consequently, robustness of such optimal plans should be studied according to different initial values. See Fard and Li (2009) and Tsai et al. (2012). Besides, V-optimality criterion and D-optimality criterion are based on large sample approximations, while ALTs are usually subject to the constraint that the available number of test units has to be small either because of high cost of the units, or availability of prototype units. In these cases, test planners may need to know the smallest possible number of units that are needed, how to choose the levels of stress, and the allocation for those units to achieve a specified precision in the ML estimators. Ma and Meeker (2010) considered the test plans when the sample size is small, and they restricted their analysis on constant-stress three-level compromise test plans. A more general method for dealing with the case of small sample size is the Bayesian approach, which can be used to combine prior information with data to provide better statistical precision. Prior information incorporated into test planning has been considered by some authors. See Zhang and Meeker (2006) and Yuan et al. (2012). Erkanli and Soyer (2000) pointed out that the Bayesian approach to the optimal design problem requires specification of three components: (i) a utility (loss) function that reflects the consequences of selecting a specific design; (ii) a probability model; (iii) a prior distribution reflecting designer's prior beliefs about all unknown quantities. Posterior variance is used as a utility function in most of the literatures about Bayesian planning. Verdinelli et al. (1993) used the Shannon information as the utility function, where the Shannon information is between marginal distribution of the data and the posterior predictive distribution. However, the method presented by Verdinelli et al. (1993) is too limited and is hard to implement. The main drawback of the criterion presented by Verdinelli et al. (1993) is that when the sample size is large enough, the optimal design will be arbitrary. See the proof in Appendix A.

The paper has two goals.

1. Propose a Bayesian criterion for planning ALT.

   We achieve this by setting the utility function as the Shannon information between prior density function and posterior density function, and defining the criterion as the expectation of such Shannon information. The optimal design is to maximize the expected Shannon information. The advantage of this Bayesian criterion is that the expected Shannon information can be interpreted as the amount of information from the observed data. Thus, an optimal test plan based on such criterion is to maximize the information from an experiment.
2. Present some algorithms to find optimal test plans.

   Large-sample approximation is first used and a simple formula is obtained for finding an optimal design. The optimal designs are the same as that based on D-optimality criterion and V-optimality criterion in some investigated examples. When the sample size is not large enough, Monte Carlo simulation algorithm is proposed. We discuss the procedures in detail for potential use.

This paper proceeds as follows. Model assumptions and the prior distributions of the parameters are given in Sections 2 and 3, respectively. Reference planning criterion is proposed in Section 4. In Section 5, we present two algorithms: large-sample approximation and Monte Carlo simulation algorithm, to find the optimal designs, and several examples are given for illustration. We compare our method with other criteria in Section 6. Finally, we conclude this paper, and discuss possible future research.

2. Model assumptions

Let T be the lifetime of a product, and suppose that T has a log-location–scale distribution with cumulative distribution function (cdf)

$$F(t|\mu,\sigma) = \Phi\left[\frac{\log(t)-\mu}{\sigma}\right], \tag{2.1}$$

where μ and σ are the location and scale parameters of log lifetime, and $\Phi(\cdot)$ is a standardized location–scale cdf. When $\Phi(\cdot) = \Phi_{\mathrm{nor}}(\cdot)$, the cdf of standard normal distribution, T is lognormal, and T follows a Weibull distribution for $\Phi(x) = 1 - \exp[-\exp(x)]$. If T follows exponential distribution, then $\sigma = 1$ and $\Phi(x) = 1 - \exp[-\exp(x)]$.

Let S denote the accelerating variable, and the most commonly used acceleration models can be denoted as

$$\mu = a + b\varphi(S), \tag{2.2}$$

where a and $b > 0$ are unknown parameters, and $\varphi(S)$ is a given decreasing function of stress level S. In particular, $\varphi(S) = 1/S$ for Arrhenius model and $\varphi(S) = -\log(S)$ for inverse power law model. σ does not depend on S, which means that the failure mechanisms are the same under different stress levels. Then S should be within a certain range, up to the highest stress level S_h, so that the failure mechanism will not change. Let S_0 be the use stress condition, and