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Predicting extinction or explosion in a Galton–Watson branching process with power series offspring distribution

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ABSTRACT

Extinction is certain in a Galton–Watson (GW) branching process if the offspring mean $\mu \leq 1$, whereas explosion is possible but not certain if $\mu > 1$. Discriminating between these two possibilities is a well-studied hypothesis-testing problem. However, deciding whether extinction or explosion will occur for the *current* realization of the process is a prediction problem. This can be formulated as a different testing problem by considering the conditional distributions of the process given extinction and explosion respectively. For power series offspring distributions, fixed-sample and sequential parametric tests are presented for the prediction problem and illustrated with data on the spread of epidemics and the populations of endangered species.

1. Introduction: the 2012 pertussis outbreak in Washington State

In 2011 the weekly numbers of new pertussis (whooping cough) cases in Washington State remained fairly constant, but in 2012 the numbers increased rapidly (Fig. 1, (CDC, 2012)). Faced with the possibility of a pandemic, the governor declared a state-wide health emergency in Week 14 and an inoculation/quarantine program was begun.

The spread of an epidemic, at least in its initial stages, can be modeled as a classical Galton–Watson (GW) branching process, cf. Section 2. The question of predicting extinction or explosion is commonly formulated as that of testing $\mu \leq 1$ (subcriticality/criticality) vs. $\mu > 1$ (supercriticality), where μ denotes the mean number of infected offspring per individual case—cf. Becker (1974), Heyde (1979), Scott (1987).[1] Guttorp and Perlman (2015) use a decision-theoretic analysis to show, however, that this problem is more complex than previous literature suggests and that the basis of a standard test procedure is somewhat dubious.

Fortunately, this testing problem usually is not the one of actual interest, because a supercritical process still may terminate with positive probability. Of more interest is the problem of predicting whether the *current realization* of a non-terminated process will terminate or explode.

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1 Basawa and Scott (1976) and Sweeting (1978) treat a related testing problem for the supercritical case.

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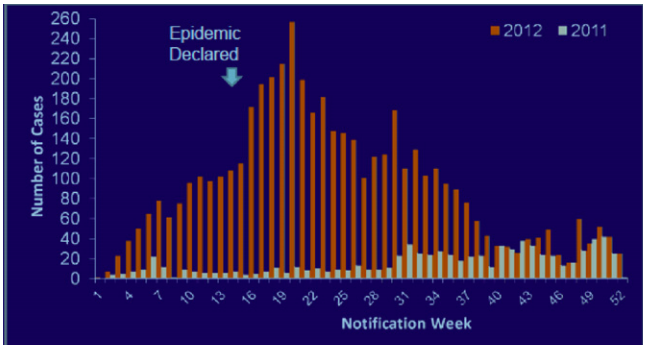

Fig. 1. Weekly counts of new pertussis cases in Washington state.

In Sections 5–6 this prediction problem is formulated as a different hypothesis-testing problem based on the conditional distributions of the process given eventual extinction and explosion respectively. Unlike the original testing problem, this prediction problem often has relatively simple solutions in the fixed-sample (Section 5) and sequential sample (Section 6) cases, the latter based on the classical Wald sequential probability ratio test (SPRT), see Section 6. Using this procedure, explosion might have been predicted for the 2012 pertussis outbreak as early as Week 3; see Example 7.2.

Like the authors noted above who treated the original testing problem, we assume a parametric model for the offspring distribution, a *power series offspring distribution (psod)*; see Section 3. The conditional distributions of a GW process given (eventual) extinction or explosion are given in Section 2, then specialized in Section 3 to the psod case. If the psod satisfies two total positivity conditions, these conditional distributions possess the stochastic monotonicity properties needed to justify our fixed-n and sequential prediction methods; see Section 4. Yaglom's (1947) well-known exponential approximation for the distribution of the population size is extended and sharpened in Sections 5.3 and 5.4.

2. Conditional processes derived from a GW branching process

The Galton–Watson branching process is a discrete-time Markov chain that describes the growth or decline of a population that reproduces by simple branching, or splitting. Applications include nuclear chain reactions, epidemics, and the population size of endangered species. The classic reference is Harris (1963, Ch. I); also see Karlin (1966), Feller (1968), Athreya and Ney (1972), Jagers (1975), Taylor and Karlin (1984), Guttorp (1991).

For each $n = 0, 1, 2, \ldots$ let X_n denote the population size at generation n; assume that $X_0 = x_0 \geq 1$ is known. At generation $n = 0$ the ith individual is replaced by a random number $\xi_i^{(1)} \stackrel{d}{=} \xi$ of first-generation offspring, where the offspring random variable (rv) $\xi \equiv \xi_{\mathbf{p}}$ has probability distribution $\mathbf{p} \equiv (p_0, p_1, p_2, \ldots)$ on $\{0, 1, 2, \ldots\}$. The ith individual in generation $n-1$ similarly is replaced by a random number $\xi_i^{(n)} \stackrel{d}{=} \xi$ of nth generation offspring independently of its siblings. Thus the population size in the nth generation satisfies

$$X_n = \xi_1^{(n)} + \cdots + \xi_{X_{n-1}}^{(n)}, \quad n \geq 1, \tag{1}$$

where $\xi_1^{(n)}, \ldots, \xi_{X_{n-1}}^{(n)}$ are iid rvs, each $\stackrel{d}{=} \xi$. We assume that each $p_k < 1$ so the process is not deterministic, and that $p_0 > 0$ so extinction is possible.

Denote the probability generating function (pgf) of the offspring distribution by

$$\phi(s) \equiv \phi_{\mathbf{p}}(s) = \mathrm{E}_{\mathbf{p}}(s^{\xi}) = \sum_{k=0}^{\infty} p_k s^k, \quad s \geq 0, \tag{2}$$

and let $1 \leq \rho \equiv \rho_{\mathbf{p}} \leq \infty$ be its radius of convergence. Note that $\phi(1) = 1$. Because $\phi(s)$ is convex and $p_1 < 1$, the equation

$$\phi(s) = s \tag{3}$$

has either one finite root or two distinct finite roots in $(0, \rho]$, one of which must be 1. If (3) has one finite root in $(0, \rho]$ denote it by $u \equiv u_{\mathbf{p}}$; if (3) has two distinct finite roots in $(0, \rho]$ denote them by $u \equiv u_{\mathbf{p}}$ and $v \equiv v_{\mathbf{p}}$, where $0 < u < v \leq \rho$.

If $x_0 = 1$, the pgf of X_n is the nth functional iterate of ϕ, denoted by ϕ_n. For $x_0 \geq 1$ the pgf of X_n is $\phi_n^{x_0} \equiv (\phi_n)^{x_0}$. Either *extinction* ($X_n = 0$ for some $n \geq 1$) or *explosion* ($X_n \to \infty$) must occur; their probabilities are u^{x_0} and $1 - u^{x_0}$ respectively.

Denote the mean of the offspring distribution by $\mu \equiv \mu_{\mathbf{p}} = \mathrm{E}(\xi)$; then $\mu = \phi'(1)$. The GW process $\mathbf{X} \equiv \mathbf{X}_{\mathbf{p}}$ and its pgf $\phi \equiv \phi_{\mathbf{p}}$ are called *subcritical (resp., critical, supercritical)* if $\mu < 1$ ($\mu = 1$, $\mu > 1$); see Fig. 2. In the subcritical case, $u = 1$ and v may or may not exist, see Section 2. In the critical case, $u = 1$ and v does not exist. In the supercritical case $0 < u < v = 1$, so both extinction and explosion occur with positive probability.