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On the superposition of overlapping Poisson processes and nonparametric estimation of their intensity function

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ABSTRACT

It is shown that, when measuring time in the Total Time on Test scale, the superposition of overlapping realizations of a nonhomogeneous Poisson process is also a Poisson process and is sufficient for inferential purposes. Hence, many nonparametric procedures which are available when only one realization is observed can be easily extended for the overlapping realizations setup. These include, for instance, the constrained maximum likelihood estimator of a monotonic intensity and bootstrap confidence bands based on Kernel estimates of the intensity. The kernel estimate proposed here performs the smoothing in the Total Time on Test scale and it is shown to behave approximately as a usual kernel estimate but with a variable bandwidth which is inversely proportional to the number of realizations at-risk. Likewise, bootstrap samples can be obtained from the single realization of the superimposed process. The methods are illustrated using a real data set consisting of the failure histories of 40 electrical power transformers.

1. Introduction

This paper is concerned with nonparametric estimation of the intensity function $\lambda(t)$ of a nonhomogeneous Poisson process (NHPP) based on the observation of K independent realizations $N_i(t)$ with multiplicative intensities $\lambda_i(t)=\lambda(t)Y_i(t)$, where $Y_i(t)\geq 0$ are known, left-continuous *at-risk* functions which vanish for $t>U_i$ $(i=1,\ldots,K)$. Our main result (cf. Proposition 1 below) states that the superposition $N_S(\tau)=\sum_{i=1}^K N_i[R^{-1}(\tau)]$ of the K original processes in the time scale $\tau=R(t)=\sum_{i=1}^K\int_0^t Y_i(t)\,dt$ is both statistically sufficient for inferences about $\lambda(\cdot)$ and it is itself an NHPP with intensity $\lambda_S(\tau)=\lambda[R^{-1}(\tau)]I(0\leq t\leq \max\{U_1,\ldots,U_K\})$. Often $Y_i(t)=I(L_i\leq t\leq U_i)$, in which case the processes $N_i(t)$ are realizations of an NHPP with intensity $\lambda(t)$ which have been observed along overlapping time periods $L_i\leq t\leq U_i$. In this setting $R(t)=\sum_{i=1}^K\int_0^t I(L_i\leq t\leq U_i)\,dt=\sum_{i=1}^K\max\{0,\min\{t-L_i,U_i-L_i\}\}$ is the familiar *Total Time on Test* (TTT) transform. In the sequel we will call $R(t)$ by this name also in the general case.

Although our main result can be proved using standard arguments in the theory of NHPPs, we have not been able to find it in the literature. We will show that it has important consequences, for it suggests that one should estimate first the intensity $\lambda_S(\tau)$ of the transformed process and then go back to the original time scale to get the estimate of $\lambda(t)=\lambda_S[R(t)]$. In other words, the many realizations setup becomes a special case of the single realization one. For instance, let t_{ij} and $\tau_{ij}=R(t_{ij})$ be respectively the event

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times of $N_i(t)$ and of $N_S(\tau)$ ($j=1,\ldots,n_i$; $i=1,\ldots,K$), so that the maximum likelihood estimate (MLE) of $\Lambda_S(\tau)=\int_0^\tau \lambda_S(s)\,ds$ is the cumulative number of events $\tilde{\Lambda}_S(\tau)=\sum_{i=1}^K\sum_{j=1}^{n_i} I(\tau_{ij}\le\tau)$ (here and below the term MLE is being used in the sense of Scholz, 1980). It follows immediately from the sufficiency of $N_S(\tau)$ and the fact that $\lambda(t)=\lambda_S[R(t)]$ that the MLE $\tilde{\Lambda}(t)$ of $\Lambda(t)=\int_0^t \lambda(u)\,du$ must satisfy, in the Riemann–Stieltjes sense, that $d\tilde{\Lambda}_S(\tau)=r(t)\,d\tilde{\Lambda}(t)$, where $r(t)=dR(t)/dt=\sum_{i=1}^K Y_i(t)$. Thus, the MLE $\tilde{\Lambda}(t)$ should be the Nelson–Aalen estimate $\tilde{\Lambda}_{NA}(t)=\sum_{i=1}^K\sum_{j=1}^{n_i} I(t_{ij}\le t)/r(t_{ij})$ (cf. Nelson, 1972, 1988; Aalen, 1978; Aalen et al., 2008; Cook and Lawless, 2007). Both finite sample and asymptotic properties of $\tilde{\Lambda}_{NA}(t)$ follow from those of $\tilde{\Lambda}_S(\tau)$ using that $N_S(\tau)$ is a Poisson process and hence has independent increments. Although these facts about $\tilde{\Lambda}_{NA}(t)$ are known, the derivation here is much simpler than what is usually found in the literature.

Besides helping to understand the relationship between existing methods for the single and the many realizations setup, Proposition 1 can be used to generalize methods which so far are available only for the single realization case. We will illustrate this fact in two ways. First, we will discuss in Section 4 how to obtain the nonparametric maximum likelihood estimate (NPMLE) of $\lambda(t)$ under a monotonicity restriction. This problem has been considered before by Boswell (1966) for a single realization and by Zielinski et al. (1993) for many overlapping realizations with $Y_i(t)=I(0\le t\le U_i)$. Our derivation here (cf. Propositions 4 and 5) is more general and, we believe, much simpler than theirs. Second, we will show in Section 5 how to obtain bootstrap confidence bands based on kernel estimates of $\lambda(t)$ using methods developed by Cowling et al. (1996) for a single realization. Unlike the usual kernel estimate of $\lambda(t)$, ours performs the smoothing in the TTT scale. We will show however that it behaves approximately as the usual kernel estimate with a variable bandwidth which is inversely proportional to the number of realizations at-risk. The main advantage of smoothing in the TTT scale is that bootstrap samples can then be obtained directly from the single realization of the superimposed process rather than from the several realizations in the original time scale, thus allowing considerable simplification from a computational point of view.

Besides Sections 4 and 5, the rest of the paper is organized as follows. To motivate the main result, the next section contains a somewhat informal discussion about the problem of obtaining the constrained NPMLE of $\lambda(t)$ in the overlapping realizations setup. The main result is stated in Section 3, where we also show that the TTT is essentially the only time transformation that preserves monotonicity of the intensities of the original and the superimposed processes. This links the TTT transform and the constrained NPMLE problem discussed in Section 4. Throughout, the proposed methodology is illustrated using a data set consisting of the failure histories of 40 electrical power transformers shown in Fig. 1(a). Particularly, we will discuss in Section 6 how to use the estimate of the intensity and associated confidence bands to obtain a nonparametric estimate of the optimal periodicity of preventive maintenance. Section 7 contains some final comments. The proofs of Propositions 1, 2 and 6 are given in the Appendix. Finally, the rest of this section reviews some additional references regarding nonparametric estimation of the intensity of an NHPP.

NHPPs play a central role in modelling situations where events occur repeatedly over time. In medical applications a realization of an NHPP is typically obtained when one considers the occurrence of events for a specific subject. On the other hand, in reliability applications one has a realization of an NHPP when considering a repairable system undergoing *minimal repair* actions at each failure, see for instance Rigdon and Basu (2000). More precisely, this means that, rather than being discarded, the system is repaired after failing and that at each failure time the system is restored to the same condition it was immediately before failing. Much of the theory regarding the use of NHPPs in these two areas overlap,

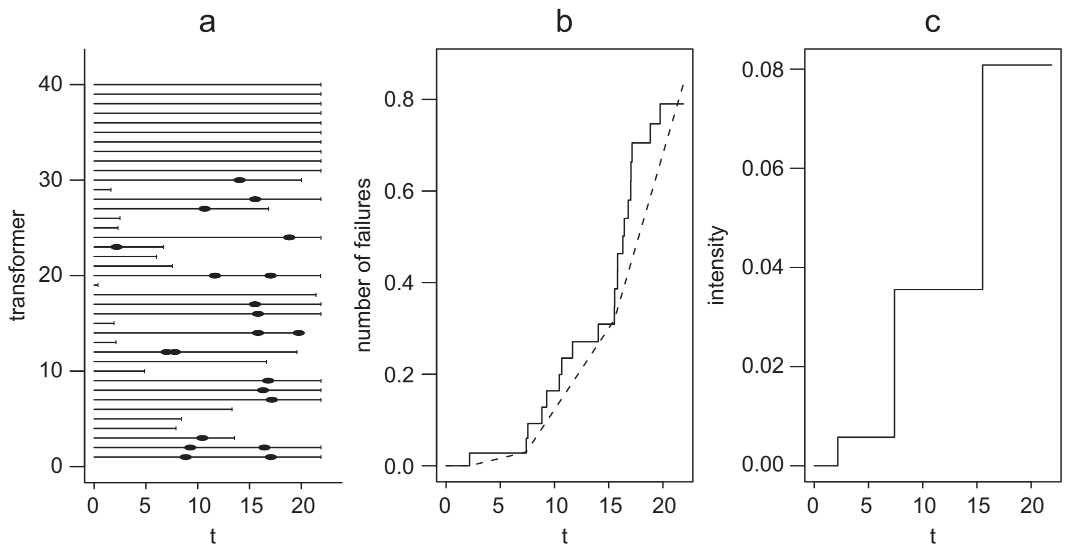

Fig. 1. (a) Power transformers data set event plot, (b) unconstrained (i.e. Nelson–Aalen, solid) and constrained (dashed) NPMLE of Λ and (c) constrained NPMLE of λ, the right derivative of the dashed line in (b). Time unit is 1000 h. Note that the constrained NPMLE of Λ intersects the unconstrained NPMLE at least twice, at about 16,500 and 19,000 h.