Journal of Statistical Planning and Inference 141 (2011) 3151–3159

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Empirical nonparametric control charts for high-quality processes

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ARTICLE INFO

Article history:
Received 17 August 2010
Received in revised form
31 January 2011
Accepted 4 April 2011
Available online 12 April 2011

Keywords:
Statistical process control
Health care monitoring
Geometric charts
Average run length
Estimated parameters
Order statistics

ABSTRACT

For attribute data with (very) small failure rates often control charts are used which decide whether to stop or to continue each time r failures have occurred, for some $r \geq 1$. Because of the small probabilities involved, such charts are very sensitive to estimation effects. This is true in particular if the underlying failure rate varies and hence the distributions involved are not geometric. Such a situation calls for a nonparametric approach, but this may require far more Phase I observations than are typically available in practice. In the present paper it is shown how this obstacle can be effectively overcome by looking not at the sum but rather at the maximum of each group of size r.

1. Introduction and motivation

High-quality processes are by now a regular phenomenon in industrial settings, due to the fact that production standards have been increasing over the last few decades. Moreover, they are in fact the norm in the area of health care monitoring, as typical failures such as malfunctioning equipment, surgical errors or recurrence of cancer, should by their very nature be quite rare. Some review papers in this latter field are Sonesson and Bock (2003), Thor et al. (2007) and Shaha (1995). Here application of control charts to improve and maintain quality is strongly advocated.

As concerns the choice of which chart to apply, many authors have argued that for the really small failure probabilities p involved it is advisable to use so-called time-between-events charts. These are typically based on waiting times until $r \geq 1$ failures have occurred. A signal then follows if the corresponding negative binomial random variable X attains a value which is judged to be too small. Many references on such negative binomial charts are given in Albers (2010), as well as a detailed analysis. In particular, it is demonstrated which choice of r is best for a given combination of underlying parameters. Moreover, Albers (2010) addresses the problem of how to deal with the fact that p is typically unknown and thus has to be estimated on the basis of a so-called Phase I sample. Simple corrections are derived which control the estimation effects involved. This estimation step is quite important: contrary to what is commonly assumed, its effects are only negligible for very large sample sizes, which in practice are typically not available.

However, besides the estimation issue, still other complications can arise. In the present context this typically concerns the underlying homogeneity assumption: each and every incoming item is supposed to have the same probability p of being defective. For some processes this assumption may be reasonable, but in medical applications patients will often show large heterogeneity. The resulting problem can be attacked by fitting a wider parametric family: in addition to the failure rate p, a second parameter is used to accommodate the degree of overdispersion. However, also this wider model

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doi:10.1016/j.jspi.2011.04.002

will achieve no perfect fit. The precise underlying mechanism remains unknown and a more flexible model is at best a better approximation of reality.

The development sketched above strongly suggests taking a next (and final) step, toward nonparametric control charts. In this way the model error, caused by estimating within a wider but still incorrect model, will nicely vanish altogether. However, the obvious catch here is the trade-off involved with respect to the stochastic error, which grows if the parametric model is widened and is prone to become unacceptably large in the nonparametric case. This is due to the fact that the false alarm rate during the in-control phase should be very small and thus extreme quantiles (for example at 0.001) need to be estimated. Hence it may seem that the nonparametric option is useful only on those rare occasions where thousands of Phase I observations are available.

Fortunately, there is a way around this obstacle. In fact, the idea is quite simple, as can be (roughly) illustrated through an example. Estimating a quantile at e.g. 0.1 rather than at 0.001 can easily be realized with a stochastic error which does remain acceptable, even for a moderate sample size. This suggests to take three observations rather than just one and to note that the probability of all of these falling below the 0.1-quantile is $(0.1)^3$, which is the desired 0.001 again. Hence judging on the basis of a – typically small – group, rather than on the basis of a single observation can do the trick. The resulting chart is truly nonparametric and both its power of detection during out-of-control and its stochastic error during in-control will be shown to be comparable to that of the standard chart, which relies on possibly dubious model assumptions.

In the present paper we shall demonstrate how this approach can be used for high-quality processes. In Section 2 we introduce the notation required in the context of the necessary background from the negative binomial approach. After the homogeneous case, the overdispersion generalization is described, as well as the further step toward a fully nonparametric approach. As the latter method typically requires too many observations, we turn in Section 3 to the new proposal based on (small) groups. In Section 4 we deal with the estimation aspects and the description of the nonparametric chart. The impact of the estimation step is analyzed and suitable corrections for the corresponding effect are proposed. For convenience, the actual application of the procedure is summarized in Section 5.

2. The negative binomial chart and generalizations

As our starting point we consider the homogeneous case, where $D_1, D_2, \ldots$, is a sequence of independent identically distributed random variables, with failure probability $P(D_1 = 1) = 1 - P(D_1 = 0) = p$ during in-control (*IC*). During the out-of-control (*OoC*) stage, this p becomes θp, for some $\theta > 1$ and the process should now be stopped as quickly as possible. First consider the negative binomial chart as discussed in Albers (2010). The 'time-between-events' approach referred to in the Introduction implies that no fixed-length blocks of D's are used. Rather we wait till the rth failure occurs, for some $r \geq 1$, and repeat this as long as the process is judged to be *IC*. Let X_i, $i = 1, 2, \ldots$ be the successive numbers of D's involved, then these X_i clearly are independent identically distributed copies of a negative binomial random variable $X_{r,p}$ such that

$$P(X_{r,p} = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \tag{2.1}$$

where $k \geq r$ is an integer. Here as well as in the sequel, we suppress indices whenever possible, and e.g. write X instead of $X_{r,p}$, unless confusion might occur.

As $\theta > 1$, a signal should result when an rth failure arrives too early. More specifically, this happens as soon as an X_i occurs which is at most equal to some well-determined lower limit $n = n_{r,p}$. In Albers (2010) it is proposed to use $F_{r,p}(n) = P(X_{r,p} \leq n) = r\alpha$ as a criterion, for some small $\alpha > 0$. Then the average run length (*ARL*) during *IC* has the same value $r/(r\alpha) = 1/\alpha$ for all r, and the negative binomial charts for various r can be compared in a fair and meaningful way. Hence $n = n_{r,p} = F_{r,p}^{-1}(r\alpha)$, the $r\alpha$th quantile of the negative binomial distribution function $F_{r,p}$. For $r = 1$, the geometric case, $F_{1,p}(n) = 1 - (1-p)^n$ and thus we have the exact result

$$n = n_{1,p} = \frac{\log(1-\alpha)}{\log(1-p)}. \tag{2.2}$$

For $r > 1$, a numerical solution is readily obtained, but in addition a transparent and accurate approximation helps to see how n depends on r, p and α. Just use that

$$F_{r,p}(n) = P(X_{r,p} \leq n) = P(Y_{n,p} \geq r) \approx P(Z_{np} \geq r), \tag{2.3}$$

where $Y_{n,p}$ is a binomial random variable with parameters n and p, while Z_{np} is a Poisson random variable with parameter $\lambda = np$. The Poisson approximation in (2.3) requires n to be large, which will be the case for $r > 1$. Hence $n \approx \lambda/p$ with λ solving $P(Z_\lambda \geq r) = r\lambda$. It is demonstrated in Albers (2010) that this λ in its turn can be approximated quite well by

$$\tilde{\lambda} = \alpha_r(1 + \zeta_r) \quad \text{with } \zeta_r = \frac{\alpha_r}{(r+1)} + \frac{\frac{1}{2}\alpha_r^2(3r+5)}{\{(r+1)^2(r+2)\}}, \tag{2.4}$$

with $\alpha_r = (r!r\alpha)^{1/r}$, for $p \leq 0.01$, $r \leq 5$ and $\alpha \leq 0.01$, which region is amply sufficient.