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Extreme eigenvalues of large dimensional quaternion sample covariance matrices

Huiqin Li, Zhidong Bai *

KLASMOE and School of Mathematics & Statistics, Northeast Normal University, Changchun, 130024, PR China

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ABSTRACT

In this paper, we investigate the almost sure limits of the largest and smallest eigenvalues of a quaternion sample covariance matrix. Suppose that $\mathbf{X}_n$ is a $p \times n$ matrix whose elements are independent quaternion variables with mean zero, variance 1 and uniformly bounded fourth moments. Denote $\mathbf{S}_n = \frac{1}{n}\mathbf{X}_n\mathbf{X}_n^*$. In this paper, we shall show that $s_{\max}(\mathbf{S}_n) = s_p(\mathbf{S}_n) \to \left(1+\sqrt{y}\right)^2$, *a.s.* and $s_{\min}(\mathbf{S}_n) \to \left(1-\sqrt{y}\right)^2$, *a.s.* as $n \to \infty$, where $y = \lim p/n$, $s_1(\mathbf{S}_n) \le \cdots \le s_p(\mathbf{S}_n)$ are the eigenvalues of $\mathbf{S}_n$, $s_{\min}(\mathbf{S}_n) = s_{p-n+1}(\mathbf{S}_n)$ when $p > n$ and $s_{\min}(\mathbf{S}_n) = s_1(\mathbf{S}_n)$ when $p \le n$. We also prove that the set of conditions are necessary for $s_{\max}(\mathbf{S}_n) \to \left(1+\sqrt{y}\right)^2$, *a.s.* when the entries of $\mathbf{X}_n$ are i. i. d.

1. Introduction

Let $\mathbf{A}$ be a $p \times p$ Hermitian matrix with eigenvalues $s_j(\mathbf{A})$, $j = 1, 2, \ldots, p$ arranged ascendingly, i.e., $s_1(\mathbf{A}) \le \cdots \le s_p(\mathbf{A})$. Then the empirical spectral distribution (ESD) of the matrix $\mathbf{A}$ is defined by

$$F^{\mathbf{A}}(x) = \frac{1}{p}\max\left\{j : s_j(\mathbf{A}) \le x\right\}.$$

If there is a sequence of random matrices whose ESD weakly converges to a limit, then the limit is said to be the LSD (Limiting Spectral Distribution) of the sequence of random matrices.

Eigenvalues of random matrix are often used in multivariate statistical analysis, such as the principal component analysis, multiple discriminant analysis, and canonical correlation analysis, etc. For example, many important statistics in multivariate statistical analysis are constructed by the eigenvalues of sample covariance matrices or those of multivariate F matrices. Moreover, they can be written as functions of integrals with respect to the ESD of sample covariance matrices or multivariate F matrices. When LSD is known, the corresponding functionals with respect to the LSD can be viewed as the population parameters and those respect to the ESD can be considered as the parameter estimators. Therefore, one may want to apply the Helly–Bray theorem to find the approximation of the statistics to their estimand. Unfortunately, the integrands are usually

* Corresponding author.
E-mail addresses: lihq118@nenu.edu.cn (H.Q. Li), baizd@nenu.edu.cn (Z.D. Bai).

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unbounded which leads to the failure of the application of the Helly–Bray theorem. Thus the limiting behavior of the extreme eigenvalues of sample covariance matrices or multivariate F matrices is of special interest.

When the underlying random variables are real and/or complex, intensive work has been done in the literature (see Geman (1980), Yin et al. (1988), Bai et al. (1988), Silverstein (1985), Bai and Yin (1993), Bai (1999), Bai and Silverstein (1998), Bai et al. (1987), among others). It is well known that the ESD of a sample covariance matrix $\mathbf{S}_n = \frac{1}{n}\mathbf{X}_n\mathbf{X}_n^*$ (the entries of $\mathbf{X}_n = \left(x_{jl}\right)_{p\times n}$ are i.i.d. real random variables with mean zero and variance σ^2) converges to the M–P (Marčenko–Pastur) law $F_y(x)$ with density

$$f_y(x) = \frac{1}{2\pi xy\sigma^2}\sqrt{(b-x)(x-a)}I_{[a,b]}(x) + I_{(1,\infty)}(y)\left(1-y^{-1}\right)\delta(x)$$

where $a = \sigma^2(1-\sqrt{y})^2$, $b = \sigma^2(1+\sqrt{y})^2$ and $y = \lim p/n \in (0,\infty)$. Here $\delta(x)$ denotes the Dirac delta function and $I_{[a,b]}(x)$ denotes the indicator function of the interval $[a,b]$. Denote the eigenvalues of $\mathbf{S}_n$ by $s_1(\mathbf{S}_n), \ldots, s_p(\mathbf{S}_n)$, arranged in ascending order. For the convergence of $s_p(\mathbf{S}_n)$, Yin et al. (1988) proved that $s_p(\mathbf{S}_n) \to \sigma^2\left(1+\sqrt{y}\right)^2$, $a.s.$ under the condition that

$$\mathrm{E}\,|y_{11}|^4 < \infty.$$

Moreover, Bai et al. (1988) showed that finite fourth moment is also necessary for the strong convergence of the largest eigenvalue. Therefore, we obtain the sufficient and necessary conditions of the strong convergence of the largest eigenvalue of $\mathbf{S}_n$. For the convergence of the smallest eigenvalue, we need to make the following declaration:

$$s_{\min}(\mathbf{S}_n) = \begin{cases} s_{\min}(\mathbf{S}_n) = s_1(\mathbf{S}_n) & p \le n, \\ s_{\min}(\mathbf{S}_n) = s_{p-n+1}(\mathbf{S}_n) & p > n. \end{cases}$$

Bai and Yin (1993) proved that

$$s_{\min}(\mathbf{S}_n) \to \sigma^2\left(1-\sqrt{y}\right)^2, \quad a.s.$$

where the underlying distribution has a zero mean and finite fourth moment. The results above were extended to the complex case in Bai (1999). When the underlying distribution is real or complex, the spectral properties of sample covariance matrices are well studied in the literature. However, due to the multiplication of quaternions is not commutative, when the entries of $\mathbf{X}_n$ are quaternion random variables, few works on the spectral properties are found in the literature unless the random variables are normality distributed, because in this case the joint density of the eigenvalues is available. In this paper, we will show that the conclusions for the quaternion sample covariance matrix remain true under only the moment conditions.

Next we introduce some notations and some basic properties about quaternions. The quaternion base can be represented by four 2×2 matrices as

$$\mathbf{e} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$ denotes the imaginary unit. Thus, a quaternion can be represented by a 2×2 complex matrix as

$$x = a\cdot\mathbf{e} + b\cdot\mathbf{i} + c\cdot\mathbf{j} + d\cdot\mathbf{k} = \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = \begin{pmatrix} \lambda & \omega \\ -\overline{\omega} & \overline{\lambda} \end{pmatrix}$$

where the coefficients a, b, c, d are real and $\lambda = a+bi$, $\omega = c+di$. The conjugate of x is defined as

$$\bar{x} = a\cdot\mathbf{e} - b\cdot\mathbf{i} - c\cdot\mathbf{j} - d\cdot\mathbf{k} = \begin{pmatrix} a-bi & -c-di \\ c-di & a+bi \end{pmatrix} = \begin{pmatrix} \overline{\lambda} & -\omega \\ \overline{\omega} & \lambda \end{pmatrix}$$

and its norm as

$$\|x\| = \sqrt{a^2+b^2+c^2+d^2} = \sqrt{|\lambda|^2+|\omega|^2}.$$

By the property of quaternions, one has

$$\det(x) = \|x\|^2. \tag{1.1}$$

Furthermore, let $\mathbf{I}_p^Q$ denote $p\times p$ quaternion identity matrix, i.e.,

$$\mathbf{I}_p^Q = \operatorname{diag}\left(\overbrace{\mathbf{e},\ldots,\mathbf{e}}^{p}\right).$$

More details can be found in Adler (1995), Finkelstein et al. (1962), Zhang (1995), Kuipers (1999), Mehta (2004), Zhang (1997), So et al. (1994). It is worth mentioning that any $p\times n$ quaternion matrix $\mathbf{X}$ can be represented by a $2p\times 2n$ complex matrix $\psi(\mathbf{X})$. Consequently, we can deal with quaternion matrices as complex matrices for convenience. Similarly, we may