Journal of Statistical Planning and Inference 143 (2013) 457–467

Contents lists available at SciVerse ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Vector random fields with compactly supported covariance matrix functions

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ARTICLE INFO

Article history:
Received 1 May 2011
Received in revised form
14 August 2012
Accepted 15 August 2012
Available online 8 September 2012

Keywords:
Covariance matrix function
Covariance tapering
Cross covariance
Direct covariance
Elliptically contoured random field
Gaussian random field
Variogram matrix function

ABSTRACT

The objective of this paper is to construct covariance matrix functions whose entries are compactly supported, and to use them as building blocks to formulate other covariance matrix functions for second-order vector stochastic processes or random fields. In terms of the scale mixture of compactly supported covariance matrix functions, we derive a class of second-order vector stochastic processes on the real line whose direct and cross covariance functions are of Pólya type. Then some second-order vector random fields in $\mathbb{R}^d$ whose direct and cross covariance functions are compactly supported are constructed by using a convolution approach and a mixture approach.

1. Introduction

Consider an m-variate stochastic process or random field $\{\mathbf{Z}(\mathbf{x}) = (Z_1(\mathbf{x}), \ldots, Z_m(\mathbf{x}))', \mathbf{x} \in \mathbb{D}\}$, which is a family of real random vectors on the same probability space, where the index set $\mathbb{D}$ could be a temporal domain like $\mathbb{Z}$, $\mathbb{R}$, or a spatial domain like $\mathbb{Z}^d$, $\mathbb{R}^d$, with d a natural number. When all of its components have second-order moments, $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ is called a second-order vector (or multivariate) random field, and its covariance matrix (function) is defined by

$$\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2) = \mathrm{E}\{(\mathbf{Z}(\mathbf{x}_1) - \mathrm{E}\mathbf{Z}(\mathbf{x}_1))(\mathbf{Z}(\mathbf{x}_2) - \mathrm{E}\mathbf{Z}(\mathbf{x}_2))'\}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}.$$

Its diagonal entry $C_{ii}(\mathbf{x}_1, \mathbf{x}_2)$, the covariance function of the ith component random field $\{Z_i(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$, is called a direct covariance (function), and its off-diagonal entry $C_{ij}(\mathbf{x}_1, \mathbf{x}_2)$ $(i \neq j)$, the covariance between the ith component random field $\{Z_i(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ and the jth component random field $\{Z_j(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$, is called a cross covariance (function), $i,j = 1,2,\ldots,m$. For properties of second-order vector random fields, see Cramer and Leadbetter (1967), Gikhman and Skorokhod (1969), Ma (2011a–d), among others. Moreover, $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ is said to be a (weakly, second-order) stationary or homogeneous random field, if its mean function $\mathrm{E}\mathbf{Z}(\mathbf{x})$, $\mathbf{x} \in \mathbb{D}$, is a constant vector, and its covariance matrix function $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ depends on the lag $\mathbf{x}_1 - \mathbf{x}_2$ only, $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbb{D}$. In such a case, we simply write $\mathbf{C}(\mathbf{x}_1 - \mathbf{x}_2)$ for $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ as usual, although it is a kind of abuse of the notation.

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http://dx.doi.org/10.1016/j.jspi.2012.08.016

We say that an m-variate random field $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ is an elliptically contoured (or spherically invariant) one, if it adopts the decomposition

$$\mathbf{Z}(\mathbf{x}) = U\mathbf{Z}_0(\mathbf{x}) + \mu(\mathbf{x}), \quad \mathbf{x} \in \mathbb{D},$$

where $\mu(\mathbf{x})$ is an m-dimensional (nonrandom) vector, U is a nonnegative random variable, and $\{\mathbf{Z}_0(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ is an m-variate zero-mean Gaussian random field and is independent of U. The term is so-called because the finite-dimensional distributions of such a random field are elliptically contoured (Fang et al., 1990). Clearly, a vector Gaussian random field is an elliptically contoured one, while an elliptically contoured one may not have second-order moments. Given a real $m \times m$ matrix function $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2), \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}$, it is shown in Ma (2011a) that there is an m-variate second-order elliptically contoured random field $\{\mathbf{Z}(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ with mean zero and with $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ as its covariance matrix if and only if $\{\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)\}' = \mathbf{C}(\mathbf{x}_2, \mathbf{x}_1)$ and the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{a}_i' \mathbf{C}(\mathbf{x}_i, \mathbf{x}_j) \mathbf{a}_j \ge 0 \qquad (1)$$

holds for every natural number n, any $\mathbf{x}_k \in \mathbb{D}$, and any $\mathbf{a}_k \in \mathbb{R}^m$, $k = 1, \ldots, n$. However, the above conditions may not be sufficient for other random fields, such as those in Matheron (1989), Emery (2010), and Ma (2011d). It would be of interest to identify other non-Gaussian random fields with the above conditions as sufficient conditions, besides the elliptically contoured one. Nevertheless, for simplicity, by a covariance matrix (function) in this paper we mean the covariance matrix (function) of a second-order elliptically contoured vector random field. Examples of elliptically contoured vector random fields include Gaussian, Student's t, hyperbolic, stable, Mittag-Leffler, Linnik, and logistic vector random fields; see Røislien and Omre (2006), Du and Ma (2011), Ma (2011a), Du et al. (2012), among others.

The objective of this paper is to construct covariance matrix functions whose entries are compactly supported, and to use them as building blocks to formulate other covariance matrix functions. By a compactly supported function in $\mathbb{R}^d$ we mean a function whose values are zero outside a compact set in $\mathbb{R}^d$. One such an example on the plane is the circular covariance function derived by Dalenius et al. (1961),

$$C(\mathbf{x}) = \begin{cases} \arccos(\|\mathbf{x}\|) - \|\mathbf{x}\|(1-\|\mathbf{x}\|^2)^{1/2}, & \|\mathbf{x}\| \le 1, \\ 0, & \|\mathbf{x}\| > 1, \mathbf{x} \in \mathbb{R}^2, \end{cases}$$

where $\|\mathbf{x}\|$ denotes the usual Euclidean norm of $\mathbf{x}$. Another example is the so-called spherical correlation model in $\mathbb{R}^3$,

$$C(\mathbf{x}) = \begin{cases} 1 - \frac{3}{2}\frac{\|\mathbf{x}\|}{\alpha} + \frac{1}{2}\frac{\|\mathbf{x}\|^3}{\alpha^3}, & \|\mathbf{x}\| \le \alpha, \\ 0, & \|\mathbf{x}\| > \alpha, \mathbf{x} \in \mathbb{R}^3, \end{cases}$$

where α is a positive constant. The third example is

$$C(\mathbf{x}) = \left(1 - \frac{\|\mathbf{x}\|}{\alpha}\right)_+^{\nu}, \quad \mathbf{x} \in \mathbb{R}^d, \qquad (2)$$

where α is a positive constant, $\nu \ge [d/2]+1$, $[x]$ denotes the largest integer that is not greater than x, and $x_+ = \max(x, 0)$, $x \in \mathbb{R}$; see Askey (1973) and Letac and Rahman (1986). More examples may be found in Wendland (1995), Wu (1995), and Fasshauer (2007), where compactly supported positive definite functions are employed for fast and efficient interpolation and approximation. This kind of positive definite functions may be also used for covariance tapering in spatial statistics, which will be addressed more specifically in the following.

Statistical inference for a univariate random field model often faces a numerical or computational challenge, involving operations on a large covariance matrix for massive spatial or spatio-temporal data. For instance, the inverse of the covariance matrix is needed when the maximum likelihood estimation method is employed to estimate covariance parameters in a Gaussian or elliptically contoured random field model; however, calculating the likelihood can be computationally infeasible for large datasets, requiring $O(n^3)$ operations. Additionally, the best linear unbiased predictor, often called a kriging predictor in geostatistics, requires the solution of a large linear system based on the covariance matrix of the observations, which gives rise to the computational hurdle when the sample size is extremely large. Covariance tapering is a useful technique to mitigate these numerical burdens; see, for instance, Kaufman et al. (2008), Zhang and Du (2008), Du et al. (2009), and Wang and Loh (2011). More specifically, the true stationary covariance function $C(\mathbf{x})$ is replaced with a tapered one, $C_{tap}(\mathbf{x}) = C(\mathbf{x})C_r(\mathbf{x})$, the direct product of the true covariance function $C(\mathbf{x})$ and a tapering function $C_r(\mathbf{x})$ that is a compactly supported correlation function taking zero beyond a certain range described by r. Therefore the resulting covariance function $C_{tap}(\mathbf{x})$ is zero outside the range r and can be executed efficiently by using well-established algorithms for sparse systems. Of course, this has to be done without sacrificing the richness of the modelling of the underlying covariance structure. To this end, Kaufman et al. (2008) established the strong consistency of the tapered likelihood estimator, Du et al. (2009) and Wang and Loh (2011) showed that appropriate tapering maintains asymptotic efficiency under a fixed-domain (infill) asymptotic framework. Obviously, the covariance matrix inverse problem becomes much more challenging when one deals with the likelihood inference, the Bayesian inference, or co-kriging for a vector Gaussian or elliptically contoured random field model. To apply the covariance tapering procedure to multivariate data, the first task is to come up with valid tapering matrix functions. However, "It seems that there exist few results on multivariate