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Robust multivariate Bayesian forecasting under functional distortions in the χ^2-metric

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ABSTRACT

This paper investigates robustness of multivariate forecasting in the Bayesian framework. The minimax approach is used to construct robust statistical procedures under deviations from hypothetical assumptions. The deviations are defined as functional distortions using the χ^2-pseudo-metric. Two cases of deviations are considered: distortions of parameter distribution and distortions of joint distribution of observations and parameters. Explicit forms for the guaranteed upper risk functional are obtained and integral equations for robust prediction statistics are given for both cases.

1. Introduction

Bayesian framework is widely used to predict future observations using a parametric model on the basis of previous ones (see, e.g., Kim et al., 2007). Due to incorporating prior information about the object under observation Bayesian approach allows improving prediction quality, especially in case of a small size sample. In practice Bayesian methods are known to be sensitive to the choice of priors. As small deviations from the assumed stochastic model cause a significant loss of optimal procedure efficiency, robustness analysis (see Huber, 1981) is necessary in order to make inferences which are reasonably reliable in the neighbourhood of the model. A detailed review of the subject of Bayesian robustness can be found in Berger (1985, 1994), Berger et al. (1996), Męczarski (1998). In this paper we explore minimax robustness of the multivariate Bayesian forecasting under functional distortions, defined using the χ^2-pseudo-metric. Similar results for the univariate model can be found in Kharin (2002).

2. Forecasting model under functional distortions

Suppose that the vector of observations $x=(x_t)_{t=1}^T \in X \subseteq \mathbb{R}^{n\times T}$ stochastically depends on θ via a hypothetical conditional probability density function (p.d.f.) $p^0(x|\theta)$, where $\theta \in \Theta \subseteq \mathbb{R}^m$ is the unobserved vector of model parameters with a hypothetical p.d.f. $\pi^0(\theta)$. The problem is to forecast the vector $y \in Y \subseteq \mathbb{R}^n$ that stochastically depends on x and θ via a hypothetical conditional p.d.f. $g^0(y|x,\theta)$.

We explore robustness of forecasting in case of the following distortions of the described hypothetical model. Suppose that the parameter vector θ is distributed according to an unknown p.d.f. $\tilde{\pi}(\theta) \in \Pi$ instead of the hypothetical $\pi^0(\theta)$. Here Π is a set

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of admissible p.d.f.s of θ:

$$\tilde{\pi}(\theta) \in \Pi = \{\tilde{\pi}(\cdot) : \rho_{\chi^2}(\pi^0(\cdot), \tilde{\pi}(\cdot)) \leq \varepsilon_+^2\}, \tag{1}$$

where the χ^2-pseudo-metric is defined as follows:

$$\rho_{\chi^2}(\pi^0(\cdot), \tilde{\pi}(\cdot)) = \int_\Theta \frac{(\pi^0(\theta) - \tilde{\pi}(\theta))^2}{\pi^0(\theta)}\, d\theta$$

and ε_+ is the distortion level. We also consider similar distortions of the joint distribution of observations and parameter vector $\nu^0(x,\theta) = p^0(x|\theta)\pi^0(\theta)$:

$$\tilde{\nu}(x,\theta) \in V = \{\tilde{\nu}(\cdot) : \rho_{\chi^2}(\nu^0(\cdot), \tilde{\nu}(\cdot)) \leq \varepsilon_+^2\},$$

$$\rho_{\chi^2}(\nu^0(\cdot), \tilde{\nu}(\cdot)) = \int_X \int_\Theta \frac{(\nu^0(x,\theta) - \tilde{\nu}(x,\theta))^2}{\nu^0(x,\theta)}\, d\theta\, dx. \tag{2}$$

The performance of a prediction statistic (p.s.) $f(\cdot) : X \to Y$ is characterized by the risk functional:

$$r(f(\cdot); \tilde{s}(\cdot)) = \int_X \int_Y \rho^2(f(x), y)\tilde{s}(x,y)\, dy\, dx, \tag{3}$$

where $\rho(\cdot,\cdot)$ is the Euclidean distance function in the space $\mathbb{R}^n$ and $\tilde{s}(\cdot,\cdot)$ is the distorted joint p.d.f. of x and y.

The guaranteed upper risk functional $r_*(\cdot)$ is used to measure the robustness of a p.s. $f(\cdot)$:

$$r_*(f(\cdot)) = \sup_{\tilde{s}(\cdot)\in S} r(f(\cdot); \tilde{s}(\cdot)), \tag{4}$$

where S is a set of admissible joint p.d.f.s $\tilde{s}(\cdot,\cdot)$.

First, we aim to find an explicit expression for the guaranteed upper risk functional under distortions (1), (2). Our second objective is to find the robust p.s. $f_*(\cdot)$:

$$r_*(f_*(\cdot)) = \inf_{f(\cdot)} r_*(f(\cdot)).$$

3. Case of distortions of $\pi^0(\theta)$

Consider distortions (1) of the hypothetical model. Denote mathematical expectations and variances calculated for the hypothetical model as $E_0\{\cdot\}$ and $D_0\{\cdot\}$, respectively. Introduce the conditional risk functional for the fixed parameter vector θ:

$$r_1(f(\cdot); \theta) = \int_X \int_Y \rho^2(f(x), y)s^0(x,y|\theta)\, dy\, dx,$$
$$s^0(x,y|\theta) = g^0(y|x,\theta)p^0(x|\theta). \tag{5}$$

Denote the conditional risk variation as $\mathring{r}_1(f(\cdot); \theta)$:

$$\mathring{r}_1(f(\cdot); \theta) = r_1(f(\cdot); \theta) - E_0\{r_1(f(\cdot); \theta)\}.$$

Introduce the critical value of the distortion level:

$$\varepsilon_\pi^*(f(\cdot)) = \frac{\sqrt{D_0\{r_1(f(\cdot); \theta)\}}}{\sup_{\theta\in\Theta} |\mathring{r}_1(f(\cdot); \theta)|}. \tag{6}$$

Theorem 1. *Let the hypothetical forecasting model be distorted according to* (1) *and for a p.s.* $f(\cdot) : X \to Y$ *the distortion level* $\varepsilon_+ \in [0, \varepsilon_\pi^*(f(\cdot))]$. *Then the guaranteed upper risk functional* (4) *can be represented as*

$$r_*(f(\cdot)) = r(f(\cdot); s^*(\cdot)), \tag{7}$$

where the extreme p.d.f. $s^*(\cdot)$ *is defined as*

$$s^*(x,y) = \int_\Theta g^0(y|x,\theta)p^0(x|\theta)\pi^*(\theta)\, d\theta,$$
$$\pi^*(\theta) = \pi^0(\theta)\left(1 + \varepsilon_+ \frac{\mathring{r}_1(f(\cdot); \theta)}{\sqrt{D_0\{r_1(f(\cdot); \theta)\}}}\right). \tag{8}$$