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Discrete distributions in the extended FGM family: The p.g.f. approach

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ABSTRACT

In this article the probability generating functions of the extended Farlie–Gumbel–Morgenstern family for discrete distributions are derived. Using the probability generating function approach various properties are examined, the expressions for probabilities, moments, and the form of the conditional distributions are obtained. Bivariate version of the geometric and Poisson distributions are used as illustrative examples. Their covariance structure and estimation of parameters for a data set are briefly discussed. A new copula is also introduced.

1. Introduction

Discrete random variables (rvs) taking non-negative integer values have received considerable attention in the literature in an effort to explain phenomena in various areas of application. For an extensive account of bivariate and multivariate distributions one can refer to the books by Kocherlakota and Kocherlakota (1992) and Johnson et al. (1997).

Models of bivariate (or multivariate) discrete distributions have been constructed by methods of convolution, random summation and mixing of distributions. The techniques commonly used for the study of various features are related to the structure of the models and the probability generating function (pgf) facilitates the derivation of various properties. Hence, (recurrence) expressions for the joint probabilities, different types of moments, and the form of the conditional distributions are obtained. However, many of these models, see Section 3.1, have restricted correlation structures imposed by the way they are constructed. Some negatively correlated bivariate Poisson distributions are discussed in Griffiths et al. (1979), and a model with Poisson marginals allowing negative correlation has been developed by Lakshminarayana et al. (1999). Nelsen (1987) constructs probability functions for dependent discrete random variables with any possible correlation value using convex linear combinations of the probability functions for the Fréchet boundary distributions.

In this article, we use standard techniques in the study of bivariate discrete distributions to obtain results on the discrete distributions that belong to the extended Farlie–Gumbel–Morgenstern (FGM) family. In Section 3 the pgf of the FGM family for discrete distributions is obtained and the distribution with geometric marginal is discussed as an alternative to a well-known

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bivariate geometric distribution. In Section 4 the pgf of the extended FGM family, in a particular case, is derived and the properties of this family of discrete distributions are given in detail. A distribution with Poisson marginals is used to fit some biological data. A new copula is also constructed in an effort to bypass difficulties in determining the possible values of a parameter.

2. The probability generating function approach

In this section we give the basic definitions and prove the necessary relations that we will use in the following sections to obtain the probability generating function of the FGM family.

Let us consider a non-negative univariate discrete random variable X with probability mass function (pmf) and cumulative distribution function (cdf) given, respectively, by

$$P(x)=\Pr[X=x] \quad \text{and} \quad F(x)=\Pr[X\le x] \quad \text{for } x=0,1,\ldots.$$

The survival function (sf) of the rv X is defined as the probability $S(x)=\Pr[X>x]$. The generating function of the pmf (pgf) and the generating function of the cdf (dgf) of the random variable X are defined as the series

$$\Pi(u)=\sum_{x=0}^{\infty}P(x)u^x \quad \text{and} \quad D(u)=\sum_{x=0}^{\infty}F(x)u^x,$$

which converge at least for $-1\le u\le 1$ and at least in the open interval $-1<u<1$, respectively.

Feller (1968, p. 265), gives a relation between the generating function of the survival function and the pgf of the rv X:

$$\sum_{x=0}^{\infty}S(x)u^x=\frac{1-\Pi(u)}{1-u} \quad \text{for } -1<u<1.$$

Since,

$$\sum_{x=0}^{\infty}S(x)u^x=\sum_{x=0}^{\infty}u^x-\sum_{x=0}^{\infty}F(x)u^x=\frac{1}{1-u}-D(u),$$

the following corollary is derived which establishes a one-to-one correspondence between the pgf and the dgf of a univariate discrete rv X.

Corollary 1. *For* $-1<u<1$

$$D(u)=\frac{\Pi(u)}{1-u}. \tag{2.1}$$

Consider now, a bivariate discrete rv $\mathbf{X}=(X_1,X_2)$ with joint pmf and joint cdf given, respectively, by

$$P_{12}(x_1,x_2)=\Pr\left[\bigcap_{i=1}^{2}(X_i=x_i)\right] \quad \text{and} \quad F_{12}(x_1,x_2)=\Pr\left[\bigcap_{i=1}^{2}(X_i\le x_i)\right],$$

for $x_i=0,1,\ldots,$ and $i=1,2$.

The pgf of the rv $\mathbf{X}$ is defined as

$$\Pi_{12}(u_1,u_2)=\sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}P_{12}(x_1,x_2)u_1^{x_1}u_2^{x_2},$$

which converges absolutely for at least $-1\le u_1,u_2\le 1$, and its dgf is defined as

$$D_{12}(u_1,u_2)=\sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}F_{12}(x_1,x_2)u_1^{x_1}u_2^{x_2},$$

which converges at least in the open square $-1<u_1,u_2<1$.

The following theorem establishes also a one-to-one correspondence between the pgf and the dgf of a bivariate discrete rv $\mathbf{X}=(X_1,X_2)$.

Theorem 1. *For* $-1<u_1,u_2<1$

$$D_{12}(u_1,u_2)=\frac{\Pi_{12}(u_1,u_2)}{(1-u_1)(1-u_2)}. \tag{2.2}$$