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On a system of nonhomogeneous components sharing a common frailty ☆

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ABSTRACT

The components of a reliability system subjected to a common random environment usually have dependent lifetimes. This paper studies the stochastic properties of such a system with lifetimes of the components following multivariate frailty models and multivariate mixed proportional reversed hazard rate (PRHR) models, respectively. Through doing stochastic comparison, we devote to throwing a new light on how the random environment affects the number of working components of a reliability system and on assessing the performance of a k-out-of-n system.

1. Introduction

For the sake of convenience and capability of computation, it is usually assumed that the lifetimes of components of a system are mutually independent when assessing the reliability. However, this is not always the case in practice. In fact, there exits a number of situations with some form of dependence among lifetimes. For example, the common random environment that affects all components of a system is a major factor inducing dependence; see, Shaked (1977) and Lindley and Singpurwalla (1986).

Consider a reliability system with components having random lifetimes $T_1,\ldots,T_n$, which operate in a common environment modeled by one random variable V. Given $V=v$, $T_1,\ldots,T_n$ are supposed to be conditionally independent. Then $(T_1,\ldots,T_n)$ has the joint survival function

$$\overline{F}(t_1,\ldots,t_n)=\int_0^\infty \prod_{i=1}^n \overline{F}_i(t_i|v)\,\mathrm{d}G_V(v), \tag{1.1}$$

where V has the distribution function G_V. Apparently, the choice of distribution function G_V depends on the actual operating environment. Thus, a basic problem is modeling the probability distribution of random environment. In this

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regard, the gamma, inverse Gaussian and Weibull distributions are popular choices for modeling the distribution of random environment because of their mathematical tractability and nice properties; see, for example, Lindley and Singpurwalla (1986), Jaisingh et al. (1993), Whitmore and Lee (1991), and Al-Mutairi and Agarwal (1997). Since there is no firm theoretical basis for choosing the probability distribution of the random environment, it is of great interest to investigate how the variation of random environment affects the performance of a system, and stochastic orderings inevitably serve as a powerful tool for such a study. Lefèvre and Malice (1989) and Ma (1999) investigated the influence of the environment on the performance of a system when the joint survival functions of the lifetimes of components are, respectively, given by

$$\int_0^\infty \prod_{i=1}^n \exp\{-\nu\lambda_i t_i\}\,\mathrm{d}G_V(\nu), \quad \lambda_i > 0 \tag{1.2}$$

and

$$\int_0^\infty \prod_{i=1}^n \overline{F}_i(\nu t_i)\,\mathrm{d}G_V(\nu).$$

In this note, we will instead consider the multivariate frailty models in (3.1), which include (1.2) as a special case, and the multivariate mixed proportional reversed hazard rate models in (4.1), respectively. We aim to discuss how the random environment affects the number of functioning components in a reliability system and the performance of a k-out-of-n system under the above two situations. Section 2 recalls some stochastic orders, and also presents some lemmas to be utilized in the sequel. Sections 3 and 4 present comparison results for multivariate frailty models and shared multivariate mixed PRHR models, respectively.

Throughout this paper, all random variables under consideration are nonnegative, "increasing" and "decreasing" mean "non-decreasing" and "non-increasing", respectively. All expectations and integrals are assumed to exist whenever they appear.

2. Preliminaries and lemmas

Let us first recall some notations of stochastic orders that will be used in this paper. Here, we are concerned with non-negative random variables which are absolutely continuous or discrete with support on integers $N_0 = \{0, 1, \ldots\}$.

Definition 2.1. For two absolutely continuous [discrete] random variables V_1 and V_2 with their probability density [mass] functions g_1 [p_1] and g_2 [p_2], distribution functions G_1 and G_2, survival functions $\overline{G}_1$ and $\overline{G}_2$, assume the ratios in the statements below are well defined. V_1 is said to be smaller than V_2 in the
(i) likelihood ratio order (denoted by $V_1 \le_{\mathrm{lr}} V_2$) if $g_2(t)/g_1(t)$ [$p_2(k)/p_1(k)$] is increasing in t[$k \in N_0$];
(ii) hazard rate order (denoted by $V_1 \le_{\mathrm{hr}} V_2$) if $\overline{G}_2(t)/\overline{G}_1(t)$ [$\overline{G}_2(k)/\overline{G}_1(k)$] is increasing in t[$k \in N_0$];
(iii) reversed hazard rate order (denoted by $V_1 \le_{\mathrm{rh}} V_2$) if $G_2(t)/G_1(t)$ [$G_2(k)/G_1(k)$] is increasing in t[$k \in N_0$];
(iv) stochastic order (denoted by $V_1 \le_{\mathrm{st}} V_2$) if $\overline{G}_2(t)[\overline{G}_2(k)] \ge \overline{G}_1(t)[\overline{G}_1(k)]$ for all t[$k \in N_0$];
(v) increasing convex [concave] order (denoted by $V_1 \le_{\mathrm{icx}} [\le_{\mathrm{icv}}] V_2$) if $\mathsf{E}[f(V_1)] \le \mathsf{E}[f(V_2)]$ for any increasing and convex [concave] f;
(vi) Laplace transform order (denoted by $V_1 \le_{\mathrm{Lt}} V_2$) if $\mathsf{E}[\exp\{-sV_1\}] \ge \mathsf{E}[\exp\{-sV_2\}]$ for all $s > 0$.

The relationship among above stochastic orders is shown in the following chain (see Shaked and Shanthikumar, 2007; Müller and Stoyan, 2002):

$$V_1 \le_{\mathrm{lr}} V_2 \Longrightarrow V_1 \le_{\mathrm{hr}} [\le_{\mathrm{rh}}] V_2 \Longrightarrow V_1 \le_{\mathrm{st}} V_2 \Longrightarrow V_1 \le_{\mathrm{icv}} [\le_{\mathrm{icx}}] V_2$$

and

$$V_1 \le_{\mathrm{icv}} V_2 \Longrightarrow V_1 \le_{\mathrm{Lt}} V_2.$$

Now, we present some lemmas which will be used in the sequel.

Lemma 2.2 (*Shaked and Shanthikumar, 2007, p. 23 and 39*). (i) *$X \le_{\mathrm{hr}} Y$ if and only if $\mathsf{E}[\alpha(X)]\mathsf{E}[\beta(Y)] \le \mathsf{E}[\alpha(Y)]\mathsf{E}[\beta(X)]$ for any α and $\beta \ge 0$ such that α/β and β are both increasing provided the finite expectations*;
(ii) *$X \le_{\mathrm{rh}} Y$ if and only if $\mathsf{E}[\alpha(X)]\mathsf{E}[\beta(Y)] \ge \mathsf{E}[\alpha(Y)]\mathsf{E}[\beta(X)]$ for any α and $\beta \ge 0$ such that α/β and β are both decreasing provided the finite expectations.*

Lemma 2.3 (*Shaked and Shanthikumar, 2007, p. 31 and 41*). *Let $X_{k:n}$ be the k-th order statistic from independent random variables $X_1, \ldots, X_n$. Then, $X_{k:n} \le_{\mathrm{hr}} [\le_{\mathrm{rh}}] X_{k+1:n}$ for $k = 1, \ldots, n-1$.*

Lemma 2.4 (*Lefèvre and Malice, 1988, Proposition 3.5(b)*). *Suppose independent Bernoulli random variables $W_1, \ldots, W_n$ with parameters $\exp\{-\lambda_1 \nu\}, \ldots, \exp\{-\lambda_n \nu\}$, respectively. Then $\mathsf{E}[f(\sum_{i=1}^n W_i)]$ is decreasing and convex with respect to ν if f is increasing and convex.*

Inspired by the proof of Lemma 2.4, we get the following analog.