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Local asymptotic normality in δ-neighborhoods of standard generalized Pareto processes

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ABSTRACT

De Haan and Pereira (2006) provided models for spatial extremes in the case of stationarity, which depend on just one parameter $\beta>0$ measuring tail dependence, and they proposed different estimators for this parameter. This framework was supplemented by Falk (2011) by establishing local asymptotic normality (LAN) of a corresponding point process of exceedances above a high multivariate threshold, yielding in particular asymptotic efficient estimators.

The estimators investigated in these papers are based on a finite set of points $t_1,\dots,t_d$, at which observations are taken. We generalize this approach in the context of functional extreme value theory (EVT). This more general framework allows estimation over some spatial parameter space, i.e., the finite set of points $t_1,\dots,t_d$ is replaced by $t\in[a,b]$. In particular, we derive efficient estimators of β based on those processes in a sample of iid processes in $C[0,1]$ which exceed a given threshold function.

1. Introduction

Suppose that the stochastic process $\boldsymbol{V}=(V_t)_{t\in[0,1]}\in C[0,1]$ is a *standard generalized Pareto process* (GPP) (Buishand et al., 2008), i.e., there exists $x_0>0$ such that

$$P(\boldsymbol{V}\le f)=1+\log(G(f)),\quad f\in\bar{E}^-[0,1],\ \|f\|_\infty\le x_0, \tag{1}$$

where $\bar{E}^-[0,1]$ is the set of those bounded functions on $[0,1]$ that attain only nonpositive values and which have a finite set of discontinuities. By G we denote the functional distribution function (df) of a *standard* max-stable process (MSP) $\boldsymbol{\eta}=(\eta_t)_{t\in[0,1]}\in C[0,1]$, i.e.,

$$G(f)=P(\boldsymbol{\eta}\le f),\quad f\in\bar{E}^-[0,1], \tag{2}$$

$P(\eta_t\le x)=\exp(x)$, $x\le 0$, $t\in[0,1]$, and $\boldsymbol{\eta}$ is *max-stable*:

$$P\left(\boldsymbol{\eta}\le\frac{f}{n}\right)^n=P(\boldsymbol{\eta}\le f),\quad f\in\bar{E}^-[0,1],\ n\in\mathbb{N}.$$

All operations on functions such as $\le$, multiplication with a constant, etc. are meant pointwise. For random functions, i.e., stochastic processes such as $\boldsymbol{V}$, $\boldsymbol{\eta}$ we use bold letters, to distinguish these from nonrandom functions such as f.

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It was observed in Aulbach et al. (2011) that the df of $\boldsymbol{\eta}$ in (2) has the representation $G(f)=\exp(-\|f\|_D)$, where $\|\cdot\|_D$ is a norm, called D-norm. It is, thus, obvious that a GPP $\boldsymbol{V}$ as in (1) is in the *functional domain of attraction* of $\boldsymbol{\eta}$: Let $\boldsymbol{V}_1,\boldsymbol{V}_2,\ldots$ be independent copies of $\boldsymbol{V}$. Then we obtain for $f\in\bar{E}^-[0,1]$

$$P\left(n\max_{1\le i\le n}\boldsymbol{V}_i\le f\right)=P\left(\max_{1\le i\le n}\boldsymbol{V}_i\le\frac{1}{n}f\right)=P\left(\boldsymbol{V}\le\frac{1}{n}f\right)^n=\left(1+\log\left(G\left(\frac{1}{n}f\right)\right)\right)^n=\left(1-\frac{1}{n}\|f\|_D\right)^n\to_{n\to\infty}\exp(-\|f\|_D)=G(f).$$

This functional domain of attraction approach was introduced in Aulbach et al. (2011). It is weaker than the approach via weak convergence of stochastic processes as developed in de Haan and Lin (2001).

De Haan and Pereira (2006) provided models for spatial extremes in the case of stationarity, which depend on just one parameter $\beta>0$ measuring tail dependence, and they proposed different estimators for this parameter. This framework was supplemented in Falk (2011) by establishing local asymptotic normality (LAN) of a corresponding point process of exceedances above a high multivariate threshold.

Precisely, it is assumed that for any $x_1,\ldots,x_d\le 0$, $d\in\mathbb{N}$,

$$P(\eta_{t_j}\le x_j,\ 1\le j\le d)=\exp\left(-\int_{-\infty}^{\infty}\max_{j\le d}(|x_j|\psi_\beta(s-t_j))ds\right),\tag{3}$$

where $\psi_\beta(s)=\beta\psi(\beta s)$ with a scale parameter $\beta>0$, and ψ is a continuous probability density on $\mathbb{R}$ with $\psi(s)=\psi(-s)>0$ and $\psi(s)$, $s\ge 0$, decreasing. Lemma 2.1 in Falk (2011) shows that the scale parameter β can be interpreted as a dependence parameter, where $\beta\to\infty$ yields asymptotically independence of $\eta_{t_1},\ldots,\eta_{t_d}$ and $\beta\to 0$ complete dependence.

In the papers by de Haan and Pereira (2006) and Falk (2011) the density ψ is known and the parameter β is estimated. The estimators investigated in these papers are based on a finite set of points $t_1<\cdots<t_d$; estimation over some interval $t\in[a,b]$ seems to be an open problem, which is the content of the present paper.

Clearly, for applications with real data, observations of a stochastic process are often taken only at a finite number of places. This raises the question whether the proposed estimator in this paper is only of theoretical interest. But, citing de Haan and Ferreira (2006, p. 293) "Infinite-dimensional extreme value theory is not just a theoretical extension of the theory to a more abstract context. It serves to solve concrete problems." And they continue with a motivating example. An alternative solution might nevertheless be to let the number of observing places go to infinity as the sample size increases. But this is work to be done in the future, with the results established here serving as finger boards.

This paper is organized as follows. In Section 2 we compile some auxiliary results and tools, in particular from functional extreme value theory (EVT). In Section 3 we introduce our estimator of β and establish its asymptotic normality under the condition that the underlying observations $\boldsymbol{V}^{(1)},\ldots,\boldsymbol{V}^{(n)}$ are independent copies of a standard GPP $\boldsymbol{V}$. Local asymptotic normality (LAN) of a corresponding point process of exceedances above a high constant threshold function is established in Section 4. This is achieved under the condition that the underlying observations are in a δ-neighborhood of a standard GPP. As an application we obtain from LAN-theory that our estimator of β is asymptotically efficient in this setup. For an account of functional EVT we refer to de Haan and Ferreira (2006); for a supplement including in particular basics of GPP we refer to Aulbach et al. (2011).

2. Auxiliary results and tools

In this section we compile several auxiliary results and tools. We start with the functional df of a standard MSP $\boldsymbol{\eta}\in C[0,1]$ whose finite dimensional marginal distributions (fidis) are given by Eq. (3).

Lemma 2.1. *We have for any* $f\in\bar{E}^-[0,1]$

$$P(\boldsymbol{\eta}\le f)=\exp\left(-\int_{-\infty}^{\infty}\sup_{t\in[0,1]}(|f(t)|\psi(s-\beta t))\,ds\right).$$

Proof. The assertion follows from the fact that a probability measure is continuous from above together with the dominated convergence theorem; note that $\int_{-\infty}^{\infty}\sup_{t\in[0,1]}\psi(s-\beta t)\,ds<\infty$. Let $Q=\{q_1,q_2,\ldots\}$ be a denumerable and dense subset of [0, 1], which contains also the set of discontinuities of f. Recall that $\boldsymbol{\eta}\in C[0,1]$. From representation (3) we obtain

$$P(\boldsymbol{\eta}\le f)=P\left(\bigcap_{n\in N}\{\eta_{q_i}\le f(q_i),\ 1\le i\le n\}\right)=\lim_{n\to\infty}P(\eta_{q_i}\le f(q_i),\ 1\le i\le n)=\lim_{n\to\infty}\exp\left(-\int_{-\infty}^{\infty}\max_{1\le i\le n}(|f(q_i)|\psi_\beta(s-q_i))\,ds\right)$$

$$=\exp\left(-\int_{-\infty}^{\infty}\lim_{n\to\infty}\left(\max_{1\le i\le n}(|f(q_i)|\psi_\beta(s-q_i))\right)ds\right)=\exp\left(-\int_{-\infty}^{\infty}\sup_{t\in[0,1]}(|f(t)|\psi(s-\beta t))\,ds\right).\qquad\square$$

The preceding result provides the functional df $P(\boldsymbol{V}\le f)=1+\log(G(f))$ of the GPP $\boldsymbol{V}$ in its upper tail.