Journal of Statistical Planning and Inference 139 (2009) 3177 – 3192

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Nonparametric control charts based on runs and Wilcoxon-type rank-sum statistics

N. Balakrishnan[a], I.S. Triantafyllou[b], M.V. Koutras[b,*]

[a] *McMaster University, Hamilton, Ontario, Canada*
[b] *University of Piraeus, Piraeus, Greece*

ARTICLE INFO

Article history:
Received 30 September 2008
Received in revised form
26 January 2009
Accepted 25 February 2009
Available online 5 March 2009

Keywords:
Average run length
Distribution-free control charts
Exceedance statistics
Lehmann alternatives
Nonparametric methods
Shewhart $\bar{X}$-chart
Statistical process control

ABSTRACT

In this article, we introduce three new distribution-free Shewhart-type control charts that exploit run and Wilcoxon-type rank-sum statistics to detect possible shifts of a monitored process. Exact formulae for the alarm rate, the run length distribution, and the average run length (*ARL*) are all derived. A key advantage of these charts is that, due to their nonparametric nature, the false alarm rate (*FAR*) and in-control run length distribution is the same for all continuous process distributions. Tables are provided for the implementation of the charts for some typical *FAR* values. Furthermore, a numerical study carried out reveals that the new charts are quite flexible and efficient in detecting shifts to Lehmann-type out-of-control situations.

1. Introduction

Statistical quality control charts were introduced in the early work of Shewhart (1926) and since then several variations of them have been proposed for monitoring continuous characteristics. Most of the control charts are distribution-based procedures in the sense that the process output is assumed to follow a specified probability distribution (usually normal); see, for example, Albers et al. (2004). However, this assumption is not often fulfilled in practice and therefore the resulting control charts may not be accurate. Recent literature have been on the development of several nonparametric methods handling efficiently hypothesis-testing problems in which no specific assumption is made about the distribution of the underlying process; for example, one may refer to the books by Gibbons and Chakraborti (2003) and Balakrishnan and Ng (2006). Albers and Kallenberg (2008) have presented a nonparametric approach for the quality control problem when adequate test observations are not observed.

In the field of statistical quality control, several nonparametric control charts, based on distribution-free hypothesis-test statistics, have been proposed. For example, Bakir (2006) has constructed a Shewhart-type control chart using a signed-rank statistic, Chakraborti and Eryilmaz (2007) have considered an alternative class of charts based on the same statistic, while Albers et al. (2006) have used a suitably modified version of a nonparametric control chart; for an overview of distribution-free control charts for continuous variables, interested readers are referred to Chakraborti et al. (2001).

In the present article, we propose three new distribution-free Shewhart-type control charts; these use specific order statistics of a reference sample to establish appropriate control limits and then exploit run or rank-based statistics defined through the test sample observations that lie between the control limits to decide whether the process is in-control or not. More specifically,

* Corresponding author. Tel.: +30 1 4142393; fax: +30 1 4142340.
E-mail address: mkoutras@unipi.gr (M.V. Koutras).

0378-3758/$ - see front matter
doi:10.1016/j.jspi.2009.02.013

the first control chart uses the maximum run length of the test sample observations in the joint sample, while the second chart takes into account the number of runs of the test sample observations whose length exceeds a pre-specified value. Finally, the third chart exploits the sum of ranks of the test sample observations that lie between the control limits.

In Section 3, we provide exact formulas for the in-control distribution of the aforementioned statistics and discuss how the calculation of the false alarm rate (*FAR*) of the new control charts can be carried out. In Section 4, out-of-control situations are investigated and an explicit formula for the alarm rate under the Lehmann-type alternatives is derived. In Section 5, an expression for the exact run length distribution is derived using a conditioning argument (see Chakraborti, 2000). Finally, in Section 6, we carry out extensive numerical computations that displays the efficacy of the new charts and their robustness features under the in-control as well as out-of-control situations.

2. Three new distribution-free control charts

Traditionally, the control limits of a distribution-free control chart are established from a reference sample drawn from a process which is in-control. Let us then denote by $X_1, X_2, \ldots, X_m$ a random sample of size m from the in-control (cumulative) distribution $F_X(x) = F(x)$ and assume that two specific order statistics, say $X_{a:m}, X_{b:m}$, are used as control limits, viz.,

$$LCL = X_{a:m}, \quad UCL = X_{b:m}$$

($1 \leqslant a < b \leqslant m$). The parameters a, b are design parameters of the chart and their determination is traditionally achieved through two different approaches. The first one requires a specific *FAR* to be achieved while the second maintains a pre-specified in-control average run length (*ARL*) value (ARL_{in}), such as 370 or 500. It is good to note here that the in-control *ARL* for a distribution-free control chart is the same for all (in-control) continuous distributions.

Suppose now test samples are drawn independently of each other (and also independently of the reference sample) and that we are interested in checking whether the process is still in-control or not. In statistical terms, if $Y_1, Y_2, \ldots, Y_n$ denotes the test sample and $F_Y(x) = G(x)$ the corresponding cumulative distribution function, our aim is to detect a possible shift in the underlying distribution from $F(x)$ to $G(x)$, i.e., to test the null hypothesis $\mathrm{H}_0: F(x) = G(x)$ against the two-sided alternative $\mathrm{H}_1: F(x) \neq G(x)$.

The test statistics used in the present article are defined in terms of

(a) runs of the Y-observations that fall within the control limits LCL, UCL,
(b) the ranks of the Y-observations that fall within the control limits LCL, UCL.

The rationale for the proposed procedures may be summarized as follows. Under the null hypothesis $\mathrm{H}_0: F = G$ (i.e., if both the reference and test samples come from the same distribution), the number of test sample observations Y_j that fall between successive X-observations should not attain "extreme" values, with extremes being determined based on the proportion n/m. With this in mind, two plausible test statistics that could be used for deciding whether the process is in-control are as follows:

- the maximum run length of Y-observations that occur between the control limits,
- the number of runs of Y-observations (between the control limits) whose length exceeds a pre-specified level k.

A third choice, different in nature from the above two, is through

- the sum of ranks (from the joint sample of X and Y observations) of the Y-observations that lie between the control limits (thereby corresponding to the well-known Wilcoxon-type rank-sum statistic).

All the aforementioned statistics may be expressed through the so-called "exceedance statistics" whose distributional properties have been discussed by a number of authors including Fligner and Wolfe (1976) and Randles and Wolfe (1979); an elaborate discussion has been provided in the monograph by Balakrishnan and Ng (2006). More specifically, let us denote by $M_i, i=1,2,\ldots,m$, the number of test sample observations Y_j that fall between the $(i-1)$-th and i-th order statistics of the X-sample (with the convention that $X_{(0)} = -\infty$). Clearly, M_i provide the lengths of runs of Y-observations between successive X-observations. Then, the three statistics mentioned above can be expressed in terms of M_i's as follows:

$$R = \max(M_{a+1}, M_{a+2}, \ldots, M_b), \quad N_k = |\{M_i : a+1 \leqslant i \leqslant b \text{ and } M_i \geqslant k\}|, \quad W = \sum_{i=a+1}^{b} W_i,$$

(k is an additional integer-valued design parameter) where W_i denotes the sum of the ranks of the Y-observations falling between $X_{(i-1)}$ and $X_{(i)}$. In order to establish a formula for the Wilcoxon-type rank-sum statistic W in terms of M_i, after noting that

$$W_i = \sum_{j=1}^{M_i} \left((i-1) + \sum_{r=1}^{i-1} M_r + j \right) = M_i \left((i-1) + \sum_{r=1}^{i-1} M_r \right) + \frac{M_i(M_i+1)}{2}$$