Journal of Statistical Planning and Inference 142 (2012) 1040–1046

Contents lists available at SciVerse ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Decomposing the complete graph into dodecahedra

P. Adams [a], D.E. Bryant [a], A.D. Forbes [b], T.S. Griggs [b,*]

[a] Department of Mathematics, University of Queensland, QLD 4072, Australia
[b] Department of Mathematics and Statistics, The Open University, Walton Hall, Milton Keynes MK7 6AA, United Kingdom

ARTICLE INFO

Article history:
Received 18 February 2011
Received in revised form
7 November 2011
Accepted 7 November 2011
Available online 15 November 2011

Keywords:
Graph decomposition
Dodecahedron

ABSTRACT

We prove that the complete graph K_v can be decomposed into dodecahedra if and only if $v \equiv 1, 16, 25$ or $40 \pmod{60}$, $v \neq 16$.

1. Introduction

The five regular Platonic solids have been known for over two millennia. In contrast, decompositions of the complete graph into copies of the graphs of the solids have a history of less than 40 years. For three of the five solids the problem is completely solved.

1. Decompositions of the complete graph K_v into tetrahedra are equivalent to Steiner systems $S(2,4,v)$. The necessary and sufficient condition is $v \equiv 1$ or $4 \pmod{12}$ (Hanani, 1975).
2. Decompositions of K_v into octahedra are equivalent to Steiner triple systems $S(2,3,v)$ which can be decomposed into Pasch configurations. The necessary and sufficient condition is $v \equiv 1$ or $9 \pmod{24}$, $v \neq 9$ (Griggs et al., 1992; Adams et al., 1994).
3. Decompositions of K_v into cubes exist if and only if $v \equiv 1$ or $16 \pmod{24}$ (Maheo, 1980; Kotzig, 1981; Bryant et al., 1994).

However, for the icosahedron and the dodecahedron, there are only partial results. For the former the admissibility condition is $v \equiv 1, 16, 21$ or $36 \pmod{60}$ and for the latter $v \equiv 1, 16, 25$ or $40 \pmod{60}$ and $v \neq 16$. Decompositions of K_v into either of these are known for $v \equiv 1 \pmod{60}$ (Adams and Bryant, 1996), but apart from this, little more is known (Adams et al., 2008), see also the dynamic survey (http://wiki.smp.uq.edu.au/G-designs/). In this paper we complete the solution for the dodecahedron. In the interests of completeness we also include a solution for the case $v \equiv 1 \pmod{60}$, but different from that given in Adams and Bryant (1996). In short, we prove the following result.

Theorem 1.1. *The complete graph K_v can be decomposed into dodecahedra if and only if $v \equiv 1, 16, 25$ or $40 \pmod{60}$, $v \neq 16$.*

Finally in this introduction, it is perhaps appropriate to mention that results are also now known for five of the 13 semi-regular Archimedean solids.

1. Decompositions of K_v into cuboctahedra exist if and only if $v \equiv 1$ or $33 \pmod{48}$ (Grannell et al., 2000).
2. Decompositions of K_v into rhombicuboctahedra exist if and only if $v \equiv 1$ or $33 \pmod{96}$ (Forbes et al., 2010).

* Corresponding author.
E-mail address: t.s.griggs@open.ac.uk (T.S. Griggs).

0378-3758/$ - see front matter
doi:10.1016/j.jspi.2011.11.001

3. Decompositions of K_v into truncated tetrahedra exist if and only if $v \equiv 1$ or 28 (mod 36) (Forbes et al., to appear).
4. Decompositions of K_v into truncated octahedra exist if and only if $v \equiv 1$ or 64 (mod 72) (Forbes et al., to appear).
5. Decompositions of K_v into truncated cubes exist if and only if $v \equiv 1$ or 64 (mod 72) (Forbes et al., to appear).

But a solution for the icosahedron remains elusive.

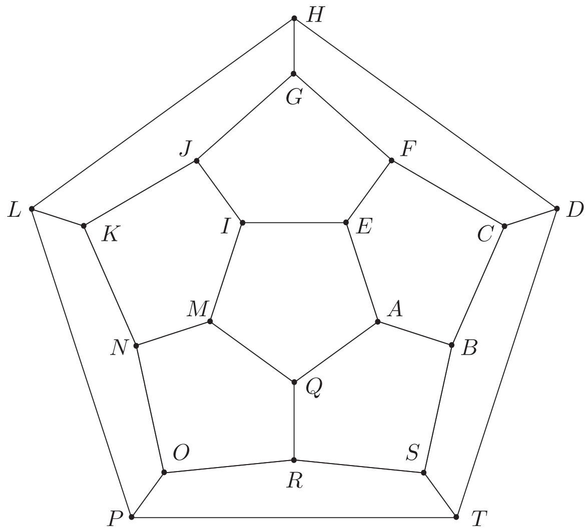
2. Building blocks

The necessity of the conditions $v \equiv 1, 16, 25$ or 40 (mod 60) and $v \neq 16$ follows by elementary counting and the fact that the dodecahedron has 20 vertices. For sufficiency, our method of proof uses a standard technique (Wilson's fundamental construction). For this we need the concept of a *group divisible design* (GDD). Recall therefore that a k-GDD of type u^t is an ordered triple (V,G,B) where V is a base set of cardinality $v=tu$, G is a partition of V into t subsets of cardinality u called *groups* and B is a family of subsets of cardinality k called *blocks* which collectively have the property that every pair of elements from *different* groups occurs in precisely one block but no pair of elements from the *same* group occurs at all. We will also need k-GDDs of type $u^t w^1$. These are defined analogously, with the base set V being of cardinality $v = tu+w$ and the partition G being into t subsets of cardinality u and one set of cardinality w.

The organization of the paper is as follows. In this section we present decompositions of various complete graphs and complete regular multipartite graphs into dodecahedra which will be needed in implementing Wilson's fundamental construction. These were all found by computer search. Then in Section 3, we present proofs of sufficiency for each of the four residue classes. As is common with proofs of this nature, certain values of v are not covered by these general constructions. These sporadic values are then dealt with in Section 4. With regard to terminology, dodecahedra will be represented by ordered 20-tuples:

$$(A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T)$$

where the co-ordinates represent vertices as in the figure below. References to the existence of various group divisible designs are to the original papers but these can also be more conveniently checked from the comprehensive listing given in the Handbook of Combinatorial Designs (Ge, 2007).



Lemma 2.1. *There exists a decomposition of the complete graph K_v into dodecahedra for $v=25$, 40, 61, 76 and 85.*

Proof.

1. K_{25}. Let the vertex set be Z_{25}. The decomposition consists of the dodecahedra

   $(0,1,2,3,4,5,7,9,6,11,8,12,10,13,14,18,15,24,19,16)$,

   $(0,6,12,1,13,3,10,8,21,4,15,22,9,23,14,2,17,7,20,11)$

   under the action of the mapping $i \mapsto i+5$ (mod 25).
2. K_{40}. Let the vertex set be $Z_{39} \cup \{\infty\}$. The decomposition consists of the dodecahedra

   $(0,1,2,3,4,6,8,10,7,11,5,15,13,20,9,18,21,14,19,28)$,