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Local power properties of some asymptotic tests in symmetric linear regression models

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A B S T R A C T

In this paper we obtain asymptotic expansions, up to order $n^{-1/2}$ and under a sequence of Pitman alternatives, for the nonnull distribution functions of the likelihood ratio, Wald, score and gradient test statistics in the class of symmetric linear regression models. This is a wide class of models which encompasses the *t* model and several other symmetric distributions with longer-than normal tails. The asymptotic distributions of all four statistics are obtained for testing a subset of regression parameters. Furthermore, in order to compare the finite-sample performance of these tests in this class of models, Monte Carlo simulations are presented. An empirical application to a real data set is considered for illustrative purposes.

1. Introduction

We say that the random variable y follows a symmetric distribution if its probability density function takes the form

$$\pi(y;\mu,\phi)=\frac{1}{\phi}\,g\left(\left(\frac{y-\mu}{\phi}\right)^2\right),\quad y\in\mathbb{R}, \tag{1}$$

where $\mu\in\mathbb{R}$ is a location parameter and $\phi>0$ is a scale parameter. The function $g:\mathbb{R}\to[0,\infty)$ is such that $\int_0^\infty g(u)\,\mathrm{d}u<\infty$ and $\int_0^\infty u^{-1/2}g(u)\,\mathrm{d}u=1$ to guarantee that $\pi(\cdot;\mu,\phi)$ is a density function. We then write $y\sim S(\mu,\phi^2)$. The function $g(\cdot)$, which is independent of y, μ and ϕ, is typically known as density generator. The probability density function of $z=(y-\mu)/\phi$ is $\pi(v;0,1)=g(v^2)$, $v\in\mathbb{R}$, i.e. $z\sim S(0,1)$. It is the standardized form of the symmetric distributions. The symmetrical family of distributions allows an extension of the normal distribution for statistical modeling of real data involving distributions with heavier and lighter tails than the ones of the normal distribution. This class of distributions is appearing with increasing frequency in the statistical literature to model several types of data containing more outlying observations than can be expected based on a normal distribution.

A number of important distributions have density function (1) and a wide range of practical applications in various fields such as engineering, biology, medicine and economics, among others. Some special cases of (1) are the following: normal, Cauchy, Student-*t*, generalized Student-*t*, type I logistic, type II logistic, generalized logistic, Kotz distribution, generalized Kotz distribution, contaminated normal, double exponential, power exponential and extended power family. These 13 distributions provide a rich source of alternative models for analyzing univariate data containing outlying observations. The modern area of research in symmetric distributions starts perhaps with the engineering applications considered by Blake and Thomas (1968) and McGraw and Wagner (1968). Applications of these distributions may also be

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found in various articles (see, for example, Chmielewski, 1981; Cambanis et al., 1981; Lange et al., 1989) whereas the properties of these distributions have been explored by Muirhead (1980, 1982), Berkane and Bentler (1986), Rao (1990) and Fang et al. (1990).

The distributions listed above provide a rich source of alternative models for analyzing univariate data containing outlying observations. However, some regularity conditions needed for the validity of our results do not hold for the Kotz, generalized Kotz and double exponential distributions. Moreover, we have not developed our results for the contaminated normal and extended power distributions. Therefore, in what follows, we restrict our attention to the following distributions: normal, Cauchy, Student-*t*, generalized Student-*t*, type I logistic, type II logistic, generalized logistic and power exponential. Additionally, all extra parameters will be considered as known or fixed. For example, the degrees of freedom ν for the Student-*t* model. The density generator of the normal, Cauchy, Student-*t*, generalized Student-*t*, type I logistic, type II logistic and power exponential are, respectively, given by $g(u)=(2\pi)^{-1/2}\exp(-u/2)$, $g(u)=\{\pi(1+u)\}^{-1}$, $g(u)=\nu^{\nu/2}B(1/2,\nu/2)^{-1}(\nu+u)^{-(\nu+1)/2}$, where $\nu>0$ and $B(\cdot,\cdot)$ is the beta function, $g(u)=s^{r/2}B(1/2,r/2)^{-1}(s+u)^{-(r+1)/2}$ $(s,r>0)$, $g(u)=c\mathrm{e}^{-u}(1+\mathrm{e}^{-u})^{-2}$, where $c\approx 1.484300029$ is the normalizing constant obtained from $\int_0^\infty u^{-1/2}g(u)\,\mathrm{d}u=1$, $g(u)=\mathrm{e}^{-\sqrt{u}}(1+\mathrm{e}^{-\sqrt{u}})^{-2}$ and $g(u)=c(k)\exp(-\frac{1}{2}u^{1/(1+k)})$, $-1<k\le 1$, where $c(k)=\Gamma(1+(k+1)/2)2^{1+(1+k)/2}$ and $\Gamma(\cdot)$ is the gamma function. For a comprehensive list of symmetric continuous distributions the reader is referred to Cordeiro et al. (2000).

The subject matter of this paper is the symmetric linear regression model which will be defined in Section 2. In this class of regression models, large-sample tests such as the likelihood ratio, Wald and Rao score tests are usually employed for testing hypotheses on the model parameters. A new criterion for testing hypotheses, referred to as the *gradient test*, was proposed in Terrell (2002). Its test statistic is very simple to compute when compared with the other three classic statistics. Here, it is worthwhile to quote Rao (2005): "The suggestion by Terrell is attractive as it is simple to compute. It would be of interest to investigate the performance of the [gradient] statistic." Also, Terrell's statistic shares the same first order asymptotic properties with the likelihood ratio, Wald and score statistics. That is, to the first order of approximation, the likelihood ratio, Wald, score and gradient statistics have the same asymptotic distributional properties either under the null hypothesis or under a sequence of Pitman alternatives, i.e. a sequence of local alternatives that shrink to the null hypothesis at a convergence rate $n^{-1/2}$. Additionally, it is known that, up to an error of order n^{-1}, the likelihood ratio, Wald, score and gradient tests have the same size properties but their local powers differ in the $n^{-1/2}$ term. Therefore, a meaningful comparison among the criteria can be performed by comparing the nonnull asymptotic expansions to order $n^{-1/2}$ of the distribution functions of these statistics under a sequence of Pitman alternatives.

The nonnull asymptotic expansions up to order $n^{-1/2}$ for the distribution functions of the likelihood ratio and Wald statistics were derived by Hayakawa (1975), while an analogous result for the score statistic was obtained by Harris and Peers (1980). The asymptotic expansion up to order $n^{-1/2}$ for the distribution functions of the gradient statistic was derived by Lemonte and Ferrari (2010). The expansions are very general, being difficult or even impossible to particularize their formulas for specific regression models. As we shall see below, we have been capable to apply their results for the class of symmetric linear regression models.

In this paper we compare the four rival tests in the class of symmetric linear models from two different points of view. First, we invoke asymptotic arguments. We then move to a finite-sample comparison, which is accomplished by means of a simulation study. Our principal aim is to help practitioners to choose among the different criteria when performing inference in symmetric linear regression models. Recently, Lemonte and Ferrari (2011) obtained the nonnull asymptotic expansions of the likelihood ratio, Wald, score and gradient statistics in Birnbaum–Saunders regression models (Rieck and Nedelman, 1991). An interesting finding is that, up to an error of order n^{-1}, the four tests have the same local power in this class of models. Their simulation study evidenced that the score and the gradient tests perform better than the likelihood ratio and Wald tests in small and moderate-sized samples and hence they concluded that the gradient test is an appealing alternative to the three classic asymptotic tests in Birnbaum–Saunders regressions. The local power of these tests in exponential family nonlinear models (Cordeiro and Paula, 1989) is presented in Lemonte (2011).

The rest of the paper is organized as follows. Section 2 presents the class of symmetric linear regression models. In Section 3 we obtain the local powers of likelihood ratio, Wald, score and gradient tests for testing a subset of regression parameters. In Section 4 we consider hypothesis testing on the parameter ϕ. Monte Carlo simulation results on the finite-sample performance of the tests are presented and discussed in Section 5. We consider an empirical application in Section 6 for illustrative purposes. Section 7 closes the paper with some concluding remarks.

2. Symmetric linear regression models

The symmetric linear regression model is defined as

$$y_l=\mu_l(\boldsymbol{\beta})+\varepsilon_l,\quad l=1,\ldots,n,$$

where $\mu_l=\mu_l(\boldsymbol{\beta})=\boldsymbol{x}_l^\top\boldsymbol{\beta}$, $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_p)^\top$ is a vector of unknown regression parameters, $\boldsymbol{x}_l^\top=(x_{l1},\ldots,x_{lp})$ contains the *i*th observation on *p* covariates and $\varepsilon_l\sim \mathrm{S}(0,\phi^2)$. We have, when they exist, that $\mathrm{E}(y_l)=\mu_l$ and $\mathrm{Var}(y_l)=\xi\phi^2$, where $\xi>0$ is a constant that may be obtained from the expected value of the radial variable or from the derivative of the characteristic function (see, for instance, Fang et al., 1990). Note that the parameter ϕ is a kind of dispersion parameter. For example, for the Student-*t* model with ν degrees of freedom we have $\xi=\nu/(\nu-2)$, for $\nu>2$.