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RCBDs with a control

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ABSTRACT

The randomized complete block designs, RCBDs, are among the most popular of block designs for comparing a set of experimental treatments. The question of this design's effectiveness when one of the treatments is a control is examined here. Optimality ranges are established for the RBCD in terms of the strength of interest in control comparisons. It is found that if the control treatment is of secondary interest, the RCBD, when not best, is typically near best. This is not so when comparisons with the control are of greater interest than those among the other treatments.

1. Introduction

A *block design* for comparing v experimental treatments is an allocation of those treatments to bk experimental units, these units having been partitioned into b sets, called blocks, of k units each. Blocks represent levels of a nuisance factor, the purpose of block stratification being to remove that noise from treatment comparisons. Having formed blocks, the quality of information the experiment will produce can vary considerably depending on the choice of design, that is, depending on the selected allocation of treatments to units within each block.

A randomized complete block design, RCBD, is a block design with blocks of size $k=v$ for which each treatment is assigned to one unit in each block. Dating all the way back to Fisher (1926, 1935), this design has been statistically justified in many ways. The RCBD is universally optimal amongst all possible designs having $k=v$ (Kiefer, 1975). RCBD analysis has a strong randomization justification; see Hinkelmann and Kempthorne (2008, Chapter 9) for many relevant references. The design is maximally robust to loss of observations, both in the sense of Morgan and Parvu (2008) and of Godolphin and Warren (2011), and is maximally robust to model inadequacies as described by Mathew and Bhaumik (1989). In addition, the RCBD offers considerable intuitive appeal for practitioners. RCBDs are arguably the most popular of block designs, finding application across the experimental spectrum, as evidenced by their inclusion in applied statistical texts in many disciplines.

The universal optimality concept mentioned above, which subsumes most conventional optimality measures, is based on the assumption that all treatment contrasts are of equal interest. Yet it is not uncommon to see the RCBD employed

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when one of the v treatments is a control. In such cases conventional optimality measures are not relevant, should interest in comparisons with the control not be on par with interest in comparisons among the other "test" treatments. This begs the question of when, and when not, the RCBD is the design of choice for experiments with a control.

Unequal interest in treatment contrasts can be formally expressed through weighted optimality measures, introduced by Morgan and Wang (2010). The needed concepts are briefly reviewed here; the standard notations employed are also defined in Morgan and Wang (2010). The class of designs under consideration, $\mathcal{D}(v,b)$, is comprised of all possible assignments of v treatments to units in b blocks of size v. The model for the observations $y_{bv\times 1}$ may be written as $y=A_d\tau+L\beta+e$ leading to the treatment contrasts information matrix $C_d=((c_{dii'}))$ specified by

$$C_d = R_d - \frac{1}{v}N_d N_d'. \tag{1}$$

In (1), R_d is the diagonal matrix of replication numbers $r_{d1},\ldots,r_{dv}$ and $N_d=(n_{dij})$ is the $v\times b$ matrix with entries n_{dij} being the number of units in block j assigned treatment i. Let Φ mapping C_d to $\Re$ be any conventional optimality function, satisfying the basic requirements specified in Eq. (2) of Morgan and Wang (2010). Select $w_1\in(0,1)$ as the *weight* assigned to the control, henceforth taking the control to be treatment 1. Form the diagonal weight matrix $W=\mathrm{diag}(w_1,w_2,\ldots,w_2)$ where $w_2=(1-w_1)/(v-1)$ is the weight assigned to each test treatment. Then the weighted information matrix is $C_{dw}=W^{-1/2}C_dW^{-1/2}$ and corresponding to Φ is the weighted optimality measure Φ_w defined by $\Phi_w(C_d)=\Phi(C_{dw})$. Let $\theta_{d1}\le\theta_{d2}\le\cdots\le\theta_{d,v-1}$ be the $v-1$ positive eigenvalues of C_{dw}. Then examples of weighted criteria are weighted A, or A_w, given by $\sum_i\theta_{di}^{-1}$, and weighted E, or E_w, given by θ_{d1}^{-1}.

Just as conventional optimality criteria are summary measures of contrast variances, weighted criteria are summary measures of weighted variances. The *weighted variance* for the contrast $c'\tau$ is $\mathrm{VAR}_w(\widehat{c'\tau})=(c'W^{-1}c)^{-1}\,\mathrm{VAR}(\widehat{c'\tau})$. Thus A_w measures average weighted variance, and E_w measures greatest weighted variance. Details are in Morgan and Wang (2010, 2011). The multiplier $(c'W^{-1}c)^{-1}$ is the weight of the contrast $c'\tau$. Selection of w_1 and thus w_2 in the current application can be made with reference to the weighted variances for elementary contrasts. The ratio of contrast weights for $\tau_i-\tau_{i'}$ relative to $\tau_1-\tau_i$ is

$$\rho=\frac{1+(v-2)w_1}{2(v-1)w_1}. \tag{2}$$

This ratio ranges from ∞ to 1/2 as w_1 ranges from 0 (no emphasis on control comparisons) to 1 (maximal emphasis on control comparisons), corresponding to the fact that in an orthogonal design, test–test comparisons can be estimated with variance arbitrarily smaller than, though no more than twice that of test–control comparisons. Results in the sections to follow are stated in terms of either w_1 or ρ according to convenience of expression; clearly the two are interchangeable. Further insight on the limiting value $\rho=1/2$ is offered in Section 3.

Weights $w_1=w_2=v^{-1}$ correspond to equal interest in all treatment contrasts ($\rho=1$), whether or not they involve the control. Weighted criteria with equal weights are conventional optimality criteria, such as those covered by universal optimality. As already stated, the RCBD is the best design in this case. Relative to equal interest, there are two other basic experimental situations: lesser interest, and greater interest, in control comparisons. The former, in which $w_1<v^{-1}$ ($\rho>1$) is selected, are termed *treatments with control* experiments, TwC for short. The latter ($\rho<1$) are *treatments versus control* experiments, or TvC for short. Examples of experiments falling into these two categories may be found in Morgan and Wang (2010), where the terminology was introduced. Together with $\rho=1$, these comprise all TC experiments with $v-1$ test treatments and one control.

The work to follow requires the two lemmas stated next. Proofs for both may be found in Morgan and Wang (2010). Let $\mathcal{P}_w$ be the class of permutation matrices preserving the weight matrix W, that is, $\mathcal{P}_w=\{P: PWP'=W\}$.

Lemma 1. *For any C_d and any $\mathcal{P}\subseteq\mathcal{P}_w$ define $\overline{C}_d=\sum_{P\in\mathcal{P}}PC_dP'/|\mathcal{P}|$. Then $\Phi_w(\overline{C}_d)\le\Phi_w(C_d)$.*

Lemma 2. *The eigenvalue θ_{d1} of the weighted information matrix C_{dw} satisfies*

(i) $\theta_{d1}\le\dfrac{c_{dii}}{w_i(1-w_i)}$ *for* $i=1,2,\ldots,v$;

(ii) $\theta_{d1}\le\dfrac{c_{dwii}+c_{dwi'i'}-2c_{dwii'}}{2-w_i-w_{i'}+2\sqrt{w_iw_{i'}}}$ *for any* $i\ne i'\in\{1,2,\ldots,v\}$.

If in Lemma 1 the set $\mathcal{P}$ is all of $\mathcal{P}_w$ the resulting $\overline{C}_d$ takes form

$$\overline{C}_d=\begin{pmatrix}\alpha_1 & \gamma 1_{1\times(v-1)}\\ \gamma 1_{(v-1)\times 1} & \alpha_2 I_{v-1}+\beta(J_{v-1}-I_{v-1})\end{pmatrix}, \tag{3}$$