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On batch queueing systems: A combinatorial approach

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ABSTRACT

We consider batch queueing systems $M/M^H/1$ and $M^H/M/1$ with catastrophes. The transient probability functions of these queueing systems are obtained by a Lattice Path Combinatorics approach that utilizes randomization and dual processes. Steady state distributions are also determined. Generalization to systems having batches of different sizes are discussed.

1. Introduction

Consider the $M/M/1$ queue with arrival rate λ and service rate μ which will have the following state rate diagram (see Fig. 1):

Let $X(t)$ and $P_{j,k}(t)$ be the queueing process at time t with $P_{j,k}(t)=P(X(t)=k|X(0)=j)$ respectively. In the classical method of solution for transient behavior as originally given by Bailey (1954), we use the difference-differential equation, probability generating function (PGF) and Laplace transform and get

$$P_{j,k}(t)=e^{-(\lambda+\mu)t}\left[\rho^{(k-j)/2}I_{k-j}\left(2\sqrt{\lambda\mu t}\right)+\rho^{(k-j-1)/2}I_{k+j+1}\left(2\sqrt{\lambda\mu t}\right)+(1-\rho)\rho^{n}\sum_{r=k+j+2}^{\infty}\rho^{-r/2}I_r\left(2\sqrt{\lambda\mu t}\right)\right] \tag{1}$$

in which $\rho=\lambda/\mu$ and $I_u(z)=\sum_{r=0}^{\infty}(z/2)^{u+2r}/r!(u+r)!$ is the modified Bessel function of the first kind (see Jain et al. (2007) p. 37). Assuming there are a virtual arrivals and b virtual service completions during time t, Champernowne (1956) provided a random walk approach which leads to the following combinatorial solution:

$$\begin{aligned}P_{j,k}(t)=e^{-(\lambda+\mu)t}&\left(\sum_{b=0}^{\infty}\frac{1}{b!(b+k-j)!}-\sum_{b=j}^{\infty}\frac{1}{(b-j)!(b+k)!}\right)\lambda^{b+k-j}\mu^{b}t^{2b+k-j}\\&+\sum_{b=j}^{\infty}\sum_{r=0}^{b-j}\left(\frac{1}{(b-j-r)!(b+k)!}-\frac{1}{(b-j-r-1)!(b+k+1)!}\right)\lambda^{b+k-j-r}\mu^{b}t^{2b+k-j-r}\end{aligned} \tag{2}$$

where $k=a+j-b$. For details see Jain et al. (2007), Section 6.3.1. The main idea is that given a and b, each event in the process is either an arrival with probability $\lambda/(\lambda+\mu)$ and a service completion with probability $\mu/(\lambda+\mu)$. Then the process

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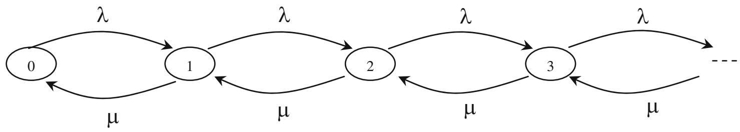

Fig. 1.

becomes a random walk with+1 step representing an arrival and -1 step representing a service completion, which is not allowed to cross 0 on the left. Eventually, the solution involves counting random walk paths subject to the stated restriction. In this paper, we derive the transient probability functions for the batch-type model in which either the arrivals are in batches of size H or the server each time serves a batch of size H if the queue size is at least H or all customers if the size is less than H. Unfortunately the counting of the corresponding random walk paths is not simple. In recent papers, Krinik and his collaborators have treated the same type of problem for quite a few models by employing randomization and dual processes. Interestingly, if we apply the same approach then we are able, in Sections 3 and 4, to obtain transient and steady state solutions of batch models by using known counting results and the theory of Markov chains.

2. Preliminaries

We state the Randomization Theorem and Duality Theorem.

Randomization Theorem (Jain et al. (2007) pages 136–137):

Suppose a Markov process on a countable state space has the transition rate matrix Q with $\sup|q_{ii}| \le c < \infty$, then the transition probability function may be written as

$$P_{i,j}(t) = e^{-ct} \sum_{n=0}^{\infty} \frac{(ct)^n}{n!} P_{i,j}^{(n)} \quad \text{for } i,j = 0,1,2,\ldots \tag{3}$$

where $P_{ij}^{(n)}$ is the n-step transition probability of the associated randomized Markov chain which has the stochastic matrix

$$P = \frac{1}{c} Q + I \tag{4}$$

as its one-step transition probability matrix, I being the identity matrix.

Actually, this result holds for sub-Markov processes; see Proposition 2.10 on page 84 of Anderson (1991). We will make use of this more general result.

Dual Processes: For a Markov process $X(t)$ having transition rate matrix Q, a dual Markov process $X^*(t)$ may exist. If it exists, the transition rate matrix of $X^*(t)$, denoted by Q^*, is defined by

$$q_{i,j}^* = \sum_{k=i}^{\infty} (q_{j,k} - q_{j-1,k}) \tag{5}$$

for $i,j = 0,1,2,\ldots$ where we assume $q_{-1,k} = 0$ for every k, see Anderson (1991), page 253.

Duality Theorem (Anderson (1991), see Proposition 4.1 and remark following this proposition on pages 251–252)

Suppose $P_{ij}(t)$ is a transition probability function having transition rate matrix Q. Define

$$P_{i,j}^*(t) = \sum_{k=i}^{\infty} [P_{j,k}(t) - P_{j-1,k}(t)] \tag{6}$$

for states $i,j = 0,1,2,\ldots$ with the convention that $P_{-1,k}(t) = 0$. Then $P_{i,j}^*(t)$ is the unique transition probability function associated with Q^* if and only if $P_{ij}(t)$ is stochastically monotone. Moreover,

$$P_{i,j}(t) = \sum_{k=0}^{i} [P_{j,k}^*(t) - P_{j+1,k}^*(t)] \quad \text{for } i,j = 1,2,\ldots \tag{7}$$

$P_{i,j}(t)$ is by definition stochastically monotone if $\sum_{j>k+1} P_{i,j}(t)$ is an increasing function of i for every fixed k and t. This says that the chance of ending up in the tail region is higher as i becomes larger. Anderson (1991) points out, page 249, that if the collection, $[q_{i,j}^*]$, satisfy the usual properties of a Q-matrix then $P_{ij}(t)$ is stochastically monotone. In fact, (6) and (7) still hold even when $P_{ij}(t)$ is a *dishonest* transition probability function corresponding to a non-conservative Q-matrix.

Lattice Path Combinatorics: If an arrival in a Markov state diagram is represented by a horizontal step to the right and a service completion in the state diagram by a vertical step upwards then a sample path on the associated randomized Markov chain is represented by a lattice path. We present below two counting results on lattice paths.