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# Consistency of $h$-mode depth

Stanislav Nagy *

*KU Leuven, Department of Mathematics and Leuven Statistics Research Centre (LStat), Belgium*
*Charles University in Prague, Department of Probability and Math. Statistics, Czech Republic*

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ABSTRACT

In this paper we establish consistency results for the sample $h$-mode depth in the general case of Banach-valued data. The rate of convergence is provided, which is linked to the rate at which the sample sequence of bandwidths converges to its population version. The robustness of the $h$-mode depth, as well as the convergence of the associated modes of the distribution, is also studied.



## 1. Introduction

Lately, in many fields of statistics, the abundance of high-, and infinite-dimensional datasets has propelled an enormous increase in the number of new statistical procedures applicable to such data. Often, the motivation for these new ideas comes from the well-established simpler procedures devised in multivariate statistics; for instance, through straightforward replacement of Euclidean spaces by general Banach spaces. However, such renditions are often burdened with lack of theoretical justification.

A prime example of a method not directly extendable to the general framework is density estimation. In the present paper we consider an estimator advanced by Cuevas et al. (2006, Section 2). In the setup of random functions (cf. Ramsay and Silverman, 2002, for functional data), the authors proposed a method emulating kernel density estimation, and used it to define a simple analogue of mode for functional data, being the point which maximizes the kernel density-like estimate over the sample space.

In Cuevas et al. (2007), the functional is dubbed the $h$-mode depth for functions and utilized successfully in the task of supervised classification. Since then, numerous authors have been considering the $h$-mode depth when constructing extensions of density-dependent concepts to functional data, or as a well performing benchmark procedure in studies involving random functions (Febrero et al., 2008; Sguera et al., 2014, 2015; Flores Díaz et al., 2014). The implementation of $h$-mode depth for functions can be found in an R package `fda.usc` (Febrero-Bande and Oviedo de la Fuente, 2012).

The main advantage of the $h$-mode depth to its competitors is its simplicity. Unlike the related concept of density-like estimation for functional data, $h$-mode depth is not defined using the small ball probabilities of the underlying random variable, and can be applied without making any distributional assumptions (see conditions (9.1)–(9.5) in Ferraty and Vieu

* Correspondence to: KU Leuven, Department of Mathematics, Statistics Section, Celestijnenlaan 200b - box 2400, 3001 Leuven, Belgium.
*E-mail address:* stanislav.nagy@wis.kuleuven.be.

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(2006), and the discussion in Section 2). Other methods of mode estimation for functional data (cf. Gasser et al., 1998; Delaigle and Hall, 2010) employ Fourier decomposition of the random functions, and their application is limited to Hilbert-valued data.

Despite the inclination of practitioners, theoretical properties of the $h$-mode depth have not yet been studied in the literature, as far as we know. The aim of the present contribution is to provide sufficient conditions for the consistency of $h$-mode depth. Here, the consistency is proved under general assumptions for Banach-valued data (cf. Araujo and Giné, 1980), and as a by-product the weak continuity of the $h$-mode depth in the probability distribution argument, as well as rates of convergence of the $h$-mode depth, are obtained. Moreover, defining the functional mode associated with the $h$-mode depth as indicated above, consistency results for the set of Banach-valued modes are shown.

By providing the necessary theoretical background, we justify the usage of the $h$-mode depth in a wide range of applications that are readily available in the literature. These include central curve estimation (Cuevas et al., 2006, 2007), outlier detection (Febrero et al., 2008; Sguera et al., 2015), homogeneity tests (Flores Díaz et al., 2014), or supervised classification (Sguera et al., 2014) for functional data.

The paper is organized as follows. In Section 2 we state the theoretical results and provide discussions on these. The proofs of the results are given in Section 3.

## 2. Consistency results for $h$-mode depth

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space on which all the random quantities are defined. For a measurable metric space $S$, $\mathcal{P}(S)$ stands for the collection of all probability measures defined on $S$ equipped with the topology of weak convergence of measures. For $P \in \mathcal{P}(S)$, $X \sim P$ means a random variable $X$ taking values in $S$ with distribution $P$. For a sequence $\{P_\nu\}_{\nu=1}^{\infty} \subset \mathcal{P}(S)$ and $P \in \mathcal{P}(S)$, the weak convergence of the former sequence to $P$ is denoted by $P_\nu \xrightarrow[\nu\to\infty]{w} P$.

In the first part of this paper, the conditions ensuring the almost complete convergence of the $h$-mode depth process are given. Recall that a sequence of real-valued random variables $\{U_\nu\}_{\nu=1}^{\infty}$ converges to a real-valued random variable $U$ almost completely if and only if

$$\sum_{\nu=1}^{\infty} \mathsf{P}\left(|U_\nu - U| > \varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{1}$$

This convergence will be denoted by $U_\nu \xrightarrow[\nu\to\infty]{\text{a.co.}} U$. The almost complete convergence is said to be of order $\{u_\nu\}_{\nu=1}^{\infty} \subset [0, \infty)$ if and only if there exists $\varepsilon_0 > 0$ such that

$$\sum_{\nu=1}^{\infty} \mathsf{P}\left(|U_\nu - U| > \varepsilon_0 u_\nu\right) < \infty,$$

written also

$$U_\nu - U = \mathcal{O}_{\text{a.co.}}\left(u_\nu\right).$$

For a more comprehensive survey of the concept of almost complete convergence, see Ferraty and Vieu (2006, Appendix A.1) and references therein. Notice that the almost complete convergence is a very powerful notion; in particular, a sequence converging almost completely converges also in the almost sure sense.

We deal with random variables taking values in a Banach space $B$ equipped with a norm $\|\cdot\|$. Let us start with the general definition of $h$-mode depth for Banach-valued data.

**Definition** (*Cuevas et al., 2006*)**.** Let $K: [0, \infty) \to [0, \infty)$ be a continuous, non-increasing function such that $K(0) > 0$, $\lim_{t\to\infty} K(t) = 0$, and let $h$ be a positive bandwidth, that is a function

$$h: \mathcal{P}(B) \to (0, \infty). \tag{2}$$

Then the $h$-mode depth of $x \in B$ with respect to the distribution $X \sim P \in \mathcal{P}(B)$ is defined as

$$D(x; P) = \frac{1}{h(P)} \mathsf{E}\left[K\left(\frac{\|x - X\|}{h(P)}\right)\right]. \tag{3}$$

The function $K$ is called the kernel function and when needed, its value at $\infty$ will be defined as $K(\infty) = \lim_{t\to\infty} K(t) = 0$.

The sample version of the $h$-mode depth is defined by replacing $P$ in (3) by the empirical measure of a random sample $X_1, \ldots, X_n$ from $P$, which will be denoted by $P_n(\omega) = P_n \in \mathcal{P}(B)$, $\omega \in \Omega$. The sample $h$-mode depth is then

$$D(x; P_n) = \frac{1}{h(P_n)n} \sum_{i=1}^{n} K\left(\frac{\|x - X_i\|}{h(P_n)}\right). \tag{4}$$