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Tests for the equality of multinomial parameters

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ABSTRACT

In this paper we face the problem of testing the equality of two or more parameters of a multinomial distribution. We develop a likelihood ratio test and we consider an asymptotically equivalent Pearson's statistic. Moreover we develop an exact and a randomized test. Relationships between these tests are then discussed. The behaviour of these tests is studied by simulations. Results from two known tests developed for less general situations are compared to ours.

1. Introduction

Many statistics have been proposed to test cell probabilities in multinomial samples. Likelihood ratio test and Pearson's statistics are often used. Typically they compare the cell frequencies to the expected values under the null hypothesis and they have asymptotically chi-squared distributions. Fienberg (1979) described the asymptotic results for goodness-of-fit statistics in categorical data problems generated by multinomial sampling schemes. Lawal (1984) analysed the same argument comparing four test statistics. Moreover, Cressie and Read (1984) studied analytically a family of test statistics for this problem which embeds most of the previous ones.

The parameters of the multinomial distribution cannot be compared each other using the previous tests. This issue was investigated by Patel (1979), who developed a test having Neyman structure to verify the equality of two parameters. Such test is uniformly most powerful unbiased (UMPU) and it is obtained by disregarding the number of cases concerning the not tested cells. To study the equality of all the multinomial parameters, Yusas (1972) proposed a test based on the maximum frequency. In this paper we propose new tests to compare the probabilities of two or more cells when nuisance parameters could be present. These tests could be used to compare preference in two or more attributes when a lot of attributes are available. Situations like the preferences of parties in proportional electoral systems, choices between treatments for the cure of one disease, research in market segmentation, are examples for which such tests could be applied.

The rest of the paper is organized as follows. In the second section we introduce the notation and develop a likelihood ratio test. In Section 3 we consider Pearson's statistic that it is asymptotically equivalent to the likelihood ratio statistic. From Pearson's statistic an exact test and a randomized test are developed in Sections 4 and 5, respectively. Section 6 is devoted to computational issues and relationships between the proposed tests. Simulation results are presented and discussed in Section 7. Conclusions are given in Section 8.

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2. Likelihood ratio test

Let π_j be the outcome probability in category j, for $j = 1, \ldots, r$. Suppose to perform n independent, identical trials; the counts $(n_1, \ldots, n_r)$ have multinomial distribution with probability mass function:

$$\frac{n!}{n_1!n_2!\cdots n_r!}\pi_1^{n_1}\pi_2^{n_2}\cdots\pi_r^{n_r} \tag{1}$$

where $n_1 + n_2 + \cdots + n_r = n$ and $\pi_1 + \pi_2 + \cdots + \pi_r = 1$.

Our goal is to test the null hypothesis $H_0 : \pi_{a_1} = \pi_{a_2} = \cdots = \pi_{a_l}$ (with $1 \leqslant a_1 < \cdots < a_l \leqslant r$; $a_i \in \mathbb{N}$ for $i = 1, \ldots, l$ and $l \in \{2, 3, \ldots, r\}$) against the alternative H_1: *at least an equality is not true*. It is easy to see that, under the null hypothesis, the maximum likelihood estimator for $\pi_{a_1} = \pi_{a_2} = \cdots = \pi_{a_l} (= \pi)$ is $\hat{\pi} = \sum_{k=1}^{l} n_{a_k}/(ln)$, while for every s different from $a_1, \ldots, a_l$ the maximum likelihood estimator for π_s is $\hat{\pi}_s = n_s/n$.

Considering the log-likelihood (L) ratio test

$$\begin{aligned} W &= 2(L(H_1) - L(H_0)) \\ &= 2\left(\sum_{k=1}^{l} n_{a_k} \ln\left(\frac{n_{a_k}}{n}\right) - \ln\left(\frac{\sum_{k=1}^{l} n_{a_k}}{ln}\right)\sum_{k=1}^{l} n_{a_k}\right) \\ &= 2\sum_{k=1}^{l} n_{a_k} \ln\left(\frac{n_{a_k} l}{\sum_{k=1}^{l} n_{a_k}}\right) \end{aligned} \tag{2}$$

we can see that the test statistic keeps into account only tested cells' counts. This statistic has an asymptotic chi-squared distribution with $l - 1$ degrees of freedom as $n \to \infty$.

3. Pearson's statistic

Pearson's statistic compares the observed values n_j with the expected values of n_j under the null hypothesis. For tested cells we expect to have about $\sum_{k=1}^{l} n_{a_k}/l$ observations in every cell so the test statistic is equal to

$$X^2 = \sum_{k=1}^{l} \frac{\left(n_{a_k} - \frac{\sum_{k=1}^{l} n_{a_k}}{l}\right)^2}{\frac{\sum_{k=1}^{l} n_{a_k}}{l}} \tag{3}$$

It is well known that also this statistic has asymptotically a chi-squared distribution with $l - 1$ degrees of freedom. The following theorem gives a relationship between likelihood ratio statistic and Pearson's statistic for this hypothesis testing.

Theorem 1. *Under* H_0 *the likelihood ratio statistic* (2) *is asymptotically equivalent to* (3) *as* $n \to \infty$.

The proof of this theorem follows the steps of an analogous one in Agresti (2002).

Proof.

$$W = 2\sum_{k=1}^{l} n_{a_k} \ln \frac{n_{a_k}}{n\hat{\pi}} = 2n\sum_{k=1}^{l} \widehat{\pi}_{a_k} \ln\left(1 + \frac{\widehat{\pi}_{a_k} - \hat{\pi}}{\hat{\pi}}\right)$$

where $\widehat{\pi}_{a_k} = n_{a_k}/n$. For $|x| < 1$ with $x = (\widehat{\pi}_{a_k} - \hat{\pi})/\hat{\pi}$ we can apply the expansion

$$\ln(1 + x) = x - x^2/2 + x^3/3 - \cdots$$

and moreover we note that x converges in probability to 0 when the model holds. Now we have

$$\begin{aligned} W &= 2n\sum_{k=1}^{l} [\widehat{\pi}_{a_k} + (\hat{\pi} - \widehat{\pi}_{a_k})]\left[\frac{\widehat{\pi}_{a_k} - \hat{\pi}}{\hat{\pi}} - \left(\frac{1}{2}\frac{(\widehat{\pi}_{a_k} - \hat{\pi})^2}{\hat{\pi}^2}\right) + \cdots\right] \\ &= 2n\sum_{k=1}^{l}\left[(\hat{\pi} - \widehat{\pi}_{a_k}) - \frac{1}{2}\frac{(\widehat{\pi}_{a_k} - \hat{\pi})^2}{\hat{\pi}} + O_p(\widehat{\pi}_{a_k} - \hat{\pi})^3\right] \\ &= n\sum_{k=1}^{l}\frac{(\widehat{\pi}_{a_k} - \hat{\pi})^2}{\hat{\pi}} + 2nO_p(n^{-3/2}) = X^2 + O_p(n^{-1/2}) = X^2 + o_p(1) \end{aligned}$$