Journal of Statistical Planning and Inference 147 (2014) 132–158

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Recursive estimation in a class of models of deformation

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ARTICLE INFO

Article history:
Received 16 January 2013
Received in revised form
4 October 2013
Accepted 16 October 2013
Available online 29 October 2013

Keywords:
Semiparametric estimation
Estimation of translation and scale parameters
Estimation of a regression function
Asymptotic properties

ABSTRACT

The paper deals with the statistical analysis of several data sets associated with shape invariant models with different translation, height and scaling parameters. We propose to estimate these parameters together with the common shape function. Our approach extends the recent work of Bercu and Fraysse to multivariate shape invariant models. We propose a very efficient Robbins–Monro procedure for the estimation of the translation parameters and we use these estimates in order to evaluate scale parameters. The common shape function is estimated by a weighted Nadaraya–Watson recursive estimator. We provide almost sure convergence and asymptotic normality for all estimators. Finally, we illustrate the convergence of our estimation procedure on simulated data as well as on real ECG data.

1. Introduction

Statistic analysis of models with periodic data is a mathematical field of great interest. Indeed, a detailed analysis of such models enables us to have a satisfactory approximation of real life phenomena. In particular, SEMOR models (Kneip and Gasser, 1988) are often used to describe a large number of phenomena such as meteorology (Vimond, 2010), road traffic (Castillo and Loubes, 2009; Gamboa et al., 2007) or children growth (Gasser and Kneip, 1995). Here, we choose to focus our attention on a particular class of these models called shape invariant models, introduced by Lawton et al. (1972). Periodic shape invariant models are semiparametric regression models with an unknown periodic shape function. In this paper, we consider several data sets associated with a common shape function, differing from each other by three parameters, a translation, a height and a scale. Formally, we are interested in the following shape invariant model:

$$Y_{i,j} = a_j f(X_i - \theta_j) + v_j + \varepsilon_{i,j}, \tag{1.1}$$

where $1 \le j \le p$ and $1 \le i \le n$, the common shape function f is periodic and the variables X_i are random, independent and of the same law. Moreover, the parameters (a_j, θ_j, v_j) belong to a subset of $\mathbb{R}^3$ which will be precised in the following. The classical approaches to estimate the different parameters of the model are to minimize the least-squares or to maximize the likelihood of the model when the law of $\varepsilon_{i,j}$ is known. Here, we propose a new recursive estimation procedure which requires very few assumptions and is really easy to implement. We study the convergence properties of our estimates when p is fixed and $n \to +\infty$, that is to say when the data are coming sequentially with respect to n. When p is not fixed and can go to infinity, the asymptotic results depend on the size of p regarding the size of n. For example, Bigot and Gadat (2010) showed that one cannot recover the correct deformation parameters when p is large and n is fixed although Bigot and Gendre (2013) pointed that it is possible when $p \lesssim n^{1/6}$. Moreover, Bigot and Charlier (2011) precised that if the size of p is too large regarding to the one of n, one cannot recover the mean pattern f with an alignment procedure. The case where the X_i

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0378-3758/$ - see front matter
http://dx.doi.org/10.1016/j.jspi.2013.10.009

are equally distributed deterministic variables have been considered especially in Gamboa et al. (2007), Trigano et al. (2011) or Vimond (2010). When $p=1$, $a_1=1$ and $v_1=0$, Bercu and Fraysse (2012) propose a recursive method to estimate the translation parameter θ_1. In this paper, we significantly extend their results as we are now able to estimate, whatever the finite value of the dimension parameter p is, the height parameter v, the translation parameter θ and the scale parameter a respectively given by

$$v=\begin{pmatrix} v_1 \\ \vdots \\ v_p \end{pmatrix}, \quad \theta=\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_p \end{pmatrix}, \quad a=\begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix}. \tag{1.2}$$

Our first goal is to estimate the translation parameter θ. The estimation of the shifts has lots of similarities with curve registration and alignment problems (Ramsay and Silverman, 2005). Analysis of ECG curves (Trigano et al., 2011) or the study of traffic data (Castillo and Loubes, 2009; Gamboa et al., 2007) fall into this framework. Gasser and Kneip (1995) propose to estimate the shifts by aligning the maxima of the curves, their position being estimated by the zeros of the derivative of a kernel estimate. In the case where $a_j=1$ and $v_j=0$, Gamboa et al. (2007) provide a semiparametric method for the estimation of the shifts. They use a Discrete Fourier Transform to transport the model (1.1) into the Fourier domain. The important contribution of Vimond (2010) generalizes this study, adding the estimation of scale and height parameters. When the parameter θ is supposed to be random, Castillo and Loubes (2009) provide sharp estimators of θ, following the approach of Dalalyan et al. (2006) in the Gaussian white noise framework. Then, they recover the unknown density of θ using a kernel density estimator. In this work, for the estimation of θ, we propose to make use of a multidimensional version of the Robbins–Monro algorithm (Robbins and Monro, 1951). Assume that one can find a function $\phi:\mathbb{R}^p\to\mathbb{R}^p$, free of the parameter θ, such that $\phi(\theta)=0$. Then, it is possible to estimate θ by the Robbins–Monro algorithm

$$\widehat{\theta}_{n+1}=\widehat{\theta}_n+\gamma_n T_{n+1} \tag{1.3}$$

where (γ_n) is a positive sequence of real numbers decreasing towards zero and (T_n) is a sequence of random vectors such that $\mathbb{E}[T_{n+1}|\mathcal{F}_n]=\phi(\widehat{\theta}_n)$ where $\mathcal{F}_n$ stands for the $\sigma-$algebra of the events occurring up to time n. Under standard conditions on the function ϕ and on the sequence (γ_n), it is well-known (Duflo, 1997) that $\widehat{\theta}_n$ tends to θ almost surely. The asymptotic normality of $\widehat{\theta}_n$ may be found in Pelletier (1998b) whereas the quadratic strong law and the law of iterated logarithm are established in Pelletier (1998a). Recently, Frikha and Menozzi (2012) established concentration bounds for Robbins–Monro algorithms. Results for randomly truncated version of the Robbins–Monro algorithm are given in Kushner and Yin (2003).

Our second goal concerns the estimation of the scale parameter a. The estimation of the scale parameters, and more particularly of the sign of the scale parameters, is very important as we can see in the numerical illustrations on daily average temperatures in Vimond (2010). Here, we obtain a strongly consistent estimate of the scale parameters a using the prior estimation of θ.

The last main theoretical part of the paper is referred to the nonparametric estimation of the common shape function f. A wide range of literature is available on nonparametric estimation of a regression function. We refer the reader to Devroye and Lugosi (2001) and Tsybakov (2004) for two complete books on density and regression function estimation. Here, we focus our attention on the Nadaraya–Watson estimator (Nadaraja, 1964; Watson, 1964) of f. More precisely, we propose to make use of a weighted recursive Nadaraya–Watson estimator (Duflo, 1997) of f which takes into account the previous estimation of v, θ and a respectively by $\widehat{v}_n$, $\widehat{\theta}_n$ and $\widehat{a}_n$. It is given, for all $x\in\mathbb{R}$, by

$$\widehat{f}_n^p(x)=\sum_{j=1}^{p}\omega_j(x)\widehat{f}_{n,j}(x) \tag{1.4}$$

with

$$\widehat{f}_{n,j}(x)=\frac{1}{\widehat{a}_{n,j}}\frac{\sum_{i=1}^{n}W_{i,j}(x)(Y_{i,j}-\widehat{v}_{i-1,j})}{\sum_{i=1}^{n}W_{i,j}(x)}$$

and

$$W_{n,j}(x)=\frac{1}{h_n}K\left(\frac{X_n-\widehat{\theta}_{n-1,j}-x}{h_n}\right),$$

where $\widehat{v}_{n,j}$, $\widehat{\theta}_{n-1,j}$ and $\widehat{a}_{n,j}$ are respectively the j-th component of $\widehat{v}_n$, $\widehat{\theta}_{n-1}$ and $\widehat{a}_n$. Moreover, the function K is a probability density function and the bandwidth (h_n) is a sequence of positive real numbers decreasing to zero. The main difficulty arising here is that we have to deal with the additional term $\widehat{\theta}_n$ inside the kernel K.

The paper is organized as follows. Section 2 presents the model and the hypothesis which are necessary to carry out our statistical analysis. Section 3 is devoted to the parametric estimation of the vector v, while Section 4 deals with our Robbins–Monro procedure for the parametric estimation of θ. Section 5 concerns the parametric estimation of the vector a. In these three sections, we establish the almost sure convergence of $\widehat{v}_n$, $\widehat{\theta}_n$ and $\widehat{a}_n$ as well as their asymptotic normality. A quadratic strong law is also provided for these three estimates. Section 6 deals with the nonparametric estimation of f. Under standard regularity assumptions on the kernel K, we prove the almost sure pointwise convergence of $\widehat{f}_n^p$ to f. In addition, we also establish the asymptotic normality of $\widehat{f}_n^p$. Section 7 contains some numerical experiments on simulated and real data,