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General dominance properties of double shrinkage estimators for ratio of positive parameters

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ABSTRACT

In estimation of ratio of variances in two normal distributions with unknown means, it has been shown in the literature that simple and crude ratio estimators based on two sample variances are dominated by shrinkage estimators using information contained in sample means. Of these, a natural double shrinkage estimator is the ratio of shrinkage estimators of variances, but its improvement over the crude ratio estimator depends on loss functions, namely, the improvement has not been established except the Stein loss function.

In this paper, this dominance property is shown for some convex loss functions including the Stein and quadratic loss functions in the general framework of distributions with positive parameters and shrinkage estimators. The resulting new finding is that the generalized Bayes estimator of the ratio of variances dominates the crude ratio estimator relative to the quadratic loss. The paper also shows that the dominance property of the double shrinkage estimator holds for estimation of the difference of variances, but it does not hold for estimation of the product and sum of variances. Finally, it is demonstrated that the double shrinkage estimators for the ratio, product, sum and differences of variances are connected to estimation of linear combinations of the positive normal means, and the dominance and non-dominance results of the double shrinkage estimators coincide with the corresponding dominance results in estimation of linear combinations of means.

1. Introduction

The estimation of a scale parameter in the presence of another nuisance parameters has been studied in the literature since Stein (1964) established the surprising result that in a normal population with unknown means, the estimator of the variance based on the sample variance with the optimal multiple, which is the best affine equivariant, is inadmissible and improved on by the Stein truncated estimator using information contained in the sample mean. Of these, Brown (1968), Brewster and Zidek (1974), Strawderman (1974) and Shinozaki (1995) developed other types of improved estimators. Since most improved estimators are smaller than or equal to the best affine equivariant, we call here them shrinkage estimators, and a class of improved and shrinkage estimators was derived by Kubokawa (1994a).

An inherited problem is the estimation of ratio $\rho = \theta_2/\theta_1$ for two scale parameters θ_1 and θ_2. A possible improvement is the single shrinkage estimators $\hat{\theta}_2^*/\hat{\theta}_1$ and $\hat{\theta}_2/\hat{\theta}_1^*$, where $\hat{\theta}_2^*$ and $1/\hat{\theta}_1^*$ are shrinkage estimators of θ_2 and $1/\theta_1$, respectively,

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improving on the crude estimators $\hat\theta_2$ and $1/\hat\theta_1$. The dominance results of such single shrinkage estimators were shown by Gelfand and Dey (1988) and Ghosh and Kundu (1996). An interesting issue is whether the single shrinkage estimators can be further improved on by a double shrinkage estimator. For the quadratic loss function $L_q(\hat\rho/\rho)=(\hat\rho/\rho-1)^2$ for estimator $\hat\rho$ of ρ, Kubokawa (1994b) demonstrated that the single shrinkage estimators can be improved on by a double shrinkage estimator of the form

$$\hat\rho^{**}=\hat\theta_2^*/\hat\theta_1+\hat\theta_2/\hat\theta_1^*-\hat\theta_2/\hat\theta_1.$$

For the Stein loss function $L_s(\hat\rho/\rho)=\hat\rho/\rho-\log(\hat\rho/\rho)-1$, Kubokawa and Srivastava (1996) and Iliopoulos and Kourouklis (1999) showed that the single shrinkage estimators can be improved on by another type of a double shrinkage estimator

$$\hat\rho^*=\hat\theta_2^*/\hat\theta_1^*.$$

Bobotas et al. (2012) developed a very nice unified theory, and clarified conditions on loss functions under which the single shrinkage estimators can be dominated by $\hat\rho^*$ and/or $\hat\rho^{**}$. We are inspired from these dominance results to raise the following queries about the double shrinkage estimators.

(I) The double shrinkage estimator $\hat\rho^*$ has a natural form, but it could not be shown that $\hat\rho^*$ dominates the single shrinkage estimators relative to the quadratic loss. Does this suggest that $\hat\rho^*$ cannot dominate the crude ratio estimator $\hat\theta_2/\hat\theta_1$? That is, we want to investigate whether the dominance property of $\hat\rho^*$ over $\hat\theta_2/\hat\theta_1$ holds for the quadratic loss.

(II) As the related problems, we can consider estimation of the product $\theta_1\theta_2$, the difference $\theta_1-\theta_2$ and the sum $\theta_1+\theta_2$. Can we extend the dominance property of double shrinkage ratio estimators to the estimation of such parameters? That is, we want to investigate whether their double shrinkage estimators dominate the corresponding crude estimators.

The objective of this paper is to reply to these queries. In Section 2, we show the dominance of $\hat\rho^*$ over the crude ratio estimator relative to some convex loss functions including the Stein and quadratic loss functions. It is noted that the dominance results hold in quite general setups as given in (A1), (A2), (A3) and (A4), namely, we do not assume any distributional assumptions except that $\hat\theta_i^*\le\hat\theta_i$ and $\hat\theta_i^*$ dominates $\hat\theta_i$ in estimation of θ_i for $i=1,2$. The dominance results will be applied in Section 3 to two sample problems of normal populations.

The query (II) is studied in Section 4. For estimation of the difference $\theta_1-\theta_2$, we can get a similar dominance result as in the case of the ratio estimation. For estimation of the product $\theta_1\theta_2$ and the sum $\theta_1+\theta_2$, however, the corresponding double shrinkage estimators cannot necessarily dominate the crude estimators. Especially, the generalized Bayes estimator $\hat\theta_1^{GB}\hat\theta_2^{GB}$ of the product never dominates the crude estimator $\hat\theta_1\hat\theta_2$ although the generalized Bayes estimator $\hat\theta_i^{GB}$ can dominate $\hat\theta_i$ in the framework of estimation of the individual parameter θ_i.

The above explanations mean that the estimation of the ratio and difference of two positive parameters has a different dominance story from the estimation of the product and sum. In Section 5, using the same arguments as in Rukhin (1992), we show that the double shrinkage estimators of the ratio, product, sum and difference are connected to estimation of the sum and difference of positive normal means. It is confirmed that the dominance and non-dominance results derived in this paper coincide with the decision-theoretic properties given by Kubokawa (2012) in the framework of estimation of the sum and difference of positive normal means.

2. General dominance results in estimation of ratio of positive parameters

Let θ_1 and θ_2 be positive unknown parameters. For $i=1,2$, let $\hat\theta_i$ and $\hat\theta_i^*$ be positive estimators of θ_i satisfying the following assumptions:

(A1) $(\hat\theta_1,\hat\theta_1^*)$ is independent of $(\hat\theta_2,\hat\theta_2^*)$.

(A2) $\hat\theta_1^*$ and $\hat\theta_2^*$ are shrinkage estimators of $\hat\theta_1$ and $\hat\theta_2$, respectively, satisfying that $\hat\theta_1^*\le\hat\theta_1$ and $\hat\theta_2^*\le\hat\theta_2$, where the strict inequalities hold with positive probabilities.

Consider the estimation of ratio of the positive parameters $\rho=\theta_2/\theta_1$. To evaluate an estimator $\tilde\rho$ of ρ, we begin by treating the risk function relative to the Stein loss function $L_s(\tilde\rho/\rho)$ for $L_s(t)=t-\log(t)-1$, namely the risk function is given by

$$R_s(\omega,\tilde\rho)=E_\omega[\tilde\rho/\rho-\log(\tilde\rho/\rho)-1]=E_\omega[\tilde\rho\theta_1/\theta_2-\log(\tilde\rho\theta_1/\theta_2)-1],$$

where ω is a collection of unknown parameters. When $\hat\theta_2$ and $1/\hat\theta_1$ are improved on by $\hat\theta_2^*$ and $1/\hat\theta_1^*$, respectively, as estimators of θ_2 and $1/\theta_1$, we want to investigate whether $\hat\rho=\hat\theta_2/\hat\theta_1$ can be improved on by the double shrinkage estimator $\hat\rho^*=\hat\theta_2^*/\hat\theta_1^*$ relative to the Stein loss. To establish the dominance property for the Stein loss function, we assume the following conditions for $\hat\theta_i$ and $\hat\theta_i^*$:

(A3) $E_\omega[\hat\theta_2/\theta_2]=1$, $E_\omega[\theta_1/\hat\theta_1]=1$ and

$$E_\omega[L_s(\hat\theta_2/\theta_2)]\ge E_\omega[L_s(\hat\theta_2^*/\theta_2)],\quad E_\omega[L_s(\theta_1/\hat\theta_1)]\ge E_\omega[L_s(\theta_1/\hat\theta_1^*)]$$

for any ω.

It is remarked that in typical cases, the best affine equivariant estimators of θ_2 and $1/\theta_1$ under Stein's loss are unbiased, namely they satisfy $E_\omega[\hat\theta_2/\theta_2]=1$ and $E_\omega[\theta_1/\hat\theta_1]=1$, so that the respective part of condition (A3) is the natural and minimal requirement for $\hat\theta_2$ and $1/\hat\theta_1$.