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On qualitative robustness of the Lotka–Nagaev estimator for the offspring mean of a supercritical Galton–Watson process

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A B S T R A C T

We characterize the sets of offspring laws on which the Lotka–Nagaev estimator for the mean of a supercritical Galton–Watson process is qualitatively robust. These are exactly the locally uniformly integrating sets of offspring laws, which may be quite large. If the corresponding global property is assumed instead, we obtain uniform robustness as well. We illustrate both results with a number of concrete examples. As a by-product of the proof we obtain that the Lotka–Nagaev estimator is [locally] uniformly weakly consistent on the respective sets of offspring laws, conditionally on non-extinction.

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1. Introduction

A Galton–Watson branching process $(Z_n) := (Z_n)_{n\in\mathbb{N}_0}$ with initial state 1 and offspring distribution μ on $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ describes the evolution of the size of a population with initial size 1, where each individual i in generation k has a random number $X_{k,i}$ of descendants drawn from μ independently of all other individuals. In other words,

$$Z_0 := 1 \quad \text{and} \quad Z_n := \sum_{i=1}^{Z_{n-1}} X_{n-1,i} \quad \text{for } n \in \mathbb{N}. \tag{1}$$

For background see, for instance, Asmussen and Hering (1983) and Athreya and Ney (1972). In this article we always assume that the mean

$$m_\mu := \sum_{k=1}^{\infty} k\,\mu[\{k\}]$$

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of the offspring distribution μ is finite. A natural estimator for the offspring mean m_μ based on observations up to time n is the Lotka–Nagaev estimator (Lotka, 1939; Nagaev, 1967) given by

$$\widehat{m}_n := \begin{cases} \dfrac{\sum_{i=1}^{Z_{n-1}} X_{n-1,i}}{Z_{n-1}} = \dfrac{Z_n}{Z_{n-1}}, & Z_{n-1} > 0, \\ 0, & Z_{n-1} = 0. \end{cases} \tag{2}$$

This estimator requires knowledge only of the last two generation sizes Z_{n-1} and Z_n. Another popular estimator is the Harris estimator $\sum_{k=1}^{n} Z_k / \sum_{k=0}^{n-1} Z_k$, which is known to be the nonparametric maximum likelihood estimator for m_μ when observing all generation sizes $Z_0, \ldots, Z_n$ (Feigin, 1977; Keiding and Lauritzen, 1978) and even when observing the entire family tree (Harris, 1948). However, in this article we restrict ourselves to the Lotka–Nagaev estimator. Note that from the point of view of applications it is often the case that the process cannot be observed for an extended period of time, such that the Lotka–Nagaev estimator is the simplest or indeed the only possible choice in these situations.

In the critical and subcritical cases, i.e. when $m_\mu \le 1$, the mean cannot be estimated consistently due to the extinction of (Z_n) with probability 1. On the other hand, in the supercritical case, i.e. when $m_\mu > 1$, the Lotka–Nagaev estimator is strongly consistent on the set of non-extinction, which can be easily shown by adapting the argument of Heyde (1970). Asymptotic normality (assuming finite variance of the offspring law μ) on the set of non-extinction was obtained by Dion (1974) among others. A discussion of further statistical properties can be found in Dion and Keiding (1978). For a recent overview of estimation in general branching processes we refer to Mitov and Yanev (2009).

The objective of the present article is to investigate the estimator $\widehat{m}_n$ for (qualitative) robustness in the supercritical case. Informally, the sequence $(\widehat{m}_n)$ is robust when a small change in μ results only in a small change of the law of the estimator $\widehat{m}_n$ uniformly in n. More precisely, given a set $\mathcal{N}$ of probability measures μ on $\mathbb{N}_0$ with $m_\mu < \infty$, the sequence of estimators $(\widehat{m}_n)$ is said to be robust on $\mathcal{N}$ if for every $\mu_1 \in \mathcal{N}$ and $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\mu_2 \in \mathcal{N}, \quad d(\mu_1, \mu_2) \le \delta \implies \rho(\mathrm{law}\{\widehat{m}_n|\mu_1\}, \mathrm{law}\{\widehat{m}_n|\mu_2\}) \le \varepsilon \text{ for all } n \in \mathbb{N}, \tag{3}$$

where d is any metric on $\mathcal{N}$ which generates the weak topology and ρ is the Prohorov metric on the set $\mathcal{M}_1^+$ of all probability measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. The sequence $(\widehat{m}_n)$ is said to be uniformly robust on $\mathcal{N}$ if δ can be chosen independently of $\mu_1 \in \mathcal{N}$. [Uniform] robustness of $(\widehat{m}_n)$ on $\mathcal{N}$ means that the set of mappings $\{\mathcal{N} \to \mathcal{M}_1^+, \mu \mapsto \mathrm{law}\{\widehat{m}_n|\mu\} : n \in \mathbb{N}\}$ is [uniformly] (d_{TV}, ρ)-equicontinuous. This definition is in line with Hampel's definition of robustness for empirical estimators in nonparametric statistical models (Cuevas, 1988; Hampel, 1971). Note, however, that our situation is *not* covered by Hampel's setting, because our estimator $\widehat{m}_n$ is not based on n i.i.d. observations. On the other hand, our setting is covered by the more general framework recently introduced in Zähle (in press). For background on robust statistics, see also Hampel et al. (1986), Huber and Ronchetti (2009) and the references cited therein.

We point out that we do *not* claim that the Lotka–Nagaev estimator is particularly robust. For a "robustification" of the Lotka–Nagaev estimator, see Stoimenova et al. (2004). We are rather interested in "how robust" the classical Lotka–Nagaev estimator is. To some extent, the degree of robustness of an estimator can be measured by the "size" of the sets $\mathcal{N}$ on which the estimator is robust; see also Zähle (in press). Intuitively, the larger the sets $\mathcal{N}$ on which the estimator is robust, the larger is the "degree" of robustness. Corollary 2.10 gives an exact specification of these sets $\mathcal{N}$ for the Lotka–Nagaev estimator. Similar investigations have recently been done by Cont et al. (2010) (see also Krätschmer et al., 2014) in the context of the empirical estimation of monetary risk measures. For instance, the empirical Value at Risk at level α (i.e., up to the sign, the empirical upper α-quantile) is robust on the set $\mathcal{N}$ of all probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with a unique α-quantile; cf. Proposition 3.5 in Cont et al. (2010).

Our main results state that the sets $\mathcal{N}$ on which the sequence $(\widehat{m}_n)$ is robust are exactly the locally uniformly integrating sets; and if a set $\mathcal{N}$ is even uniformly integrating and satisfies $\inf_{\mu \in \mathcal{N}} m_\mu > 1$, then $(\widehat{m}_n)$ is even uniformly robust on it. Uniformly integrating for a set $\mathcal{N}$ means just that any set of random variables $\{Y \sim \mu : \mu \in \mathcal{N}\}$ is uniformly integrable. This property is just a tiny bit stronger than finiteness of $\sup_{\mu \in \mathcal{N}} m_\mu$; see Remark 2.3. Locally uniformly integrating means that every weakly convergent subsequence in $\mathcal{N}$ is uniformly integrating.

In Section 2 we also provide various examples of (parametric) sets $\mathcal{N}$ that are [locally] uniformly integrable. We illustrate the implied robustness statements in the context of estimating a parameter (via estimating the mean) that is either slightly perturbed or belongs to a model that is slightly misspecified. In both situations [uniform] robustness yields that the distribution of the estimator is largely unaffected.

2. Main results and discussion

For the exact formulation of our main results we have to define the Galton–Watson process as a sort of canonical process. More precisely, let $(Z_n) := (Z_n)_{n \in \mathbb{N}_0}$ be given by (1) with $(X_{k,i}) := (X_{k,i})_{(k,i) \in \mathbb{N}_0 \times \mathbb{N}}$ the coordinate process on

$$(\Omega, \mathcal{F}) := (\mathbb{N}_0^{\mathbb{N}_0 \times \mathbb{N}}, \mathfrak{P}(\mathbb{N}_0)^{\otimes(\mathbb{N}_0 \times \mathbb{N})})$$

(with $\mathfrak{P}$ denoting the set of all subsets) under the product law

$$\mathbb{P}^\mu := \mu^{\otimes(\mathbb{N}_0 \times \mathbb{N})}.$$

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