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The multivariate alpha-power model

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ABSTRACT

The main object of this paper is to propose a multivariate extension to the alpha-power model which is an alternative to the multivariate skew-normal model (Arellano-Valle and Azzalini, 2008). It also extends the power-normal model discussed in Gupta and Gupta (2008) by making it more flexible. Inference is dealt with by using the likelihood approach and a pseudo-likelihood approach based on conditional distributions which, although slightly less efficient, is simpler to implement. An application to a real data set is used to demonstrate the usefulness of the extension.

1. Introduction

The literature on families of flexible distributions has experienced a great increase in the last three or four decades. Some early results include Birnbaum (1950), Lehmann (1953), Roberts (1966), O'Hagan and Leonard (1976), among others. More recently, Azzalini (1985), Fernández and Steel (1998), Mudholkar and Hutson (2000), Gupta et al. (2002), Arellano-Valle et al. (2004, 2005) and Gómez et al. (2007), represent some of the important references on the subject.

Azzalini (1985), has considered a general structure for asymmetric distributions in the univariate situation, which is given by

$$\varphi(z;\lambda) = 2f(z)G(\lambda z), \quad z,\lambda \in \mathbb{R}, \tag{1}$$

where f is a symmetric (about zero) probability density function and G is the distribution function of a density which is symmetric about zero. Properties of the density (1) have been extensively studied by Azzalini (1986), Henze (1986), Chiogna (1997), Pewsey (2000) and Gómez et al. (2007), among others. The special case where $f=\phi$ and $G=\Phi$, the density and distribution function of standard normal, respectively, corresponds to the standard skew-normal distribution with density function given by

$$\varphi(z;\lambda) = 2\phi(z)\Phi(\lambda z), \quad z \in \mathbb{R}, \tag{2}$$

denoted by $Z \sim SN(\lambda)$.

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In this paper, a multivariate asymmetric model called the multivariate alpha-power normal model is considered which extends the univariate alpha-power family studied in Pewsey et al. (2012). A particular case, the power-normal model, is studied in Gupta and Gupta (2008). It was shown in Pewsey et al. (2012) that the Fisher information for the alpha-power model is nonsingular. It is an attractive alternative to the skew-normal model which can present estimation difficulties such as a singular information matrix and an unbounded likelihood function (Azzalini, 1985; Arellano-Valle and Azzalini, 2008; Gupta and Gupta, 2008). The multivariate distribution is constructed by using results in Arnold and Strauss (1988, 1991a,b). Maximum likelihood estimation is discussed which can be complicated to implement given the complex nature of the normalizing constant. As an alternative, the pseudo-likelihood approach can be implemented, which although less efficient than the ordinary likelihood is simpler to implement.

The paper is organized as follows. In Section 2, for the sake of completeness, we define the alpha-power model and investigate some of its properties. In Section 3 the bivariate alpha-power model is constructed by using results in Arnold et al. (1992, 1999, 2001,2002). Measures of dependency are investigated. In Section 4 an extension is considered for the p-variate model. Given that likelihood estimation depends on the computation of complicated normalizing constants, a pseudo-likelihood function is constructed based on conditional distributions. In Section 5, we consider an application to the bivariate situation which suggests that the new model can substantially improve upon the bivariate normal distribution. General comments are presented in Section 6 and the paper concludes with an appendix on the main mathematical derivations.

2. The univariate alpha-power distribution

Lehmann (1953) studied the family of distributions with a general distribution function given by

$$\mathcal{F}_F(z;\alpha)=\{F(z)\}^{\alpha},\quad z\in\mathbb{R}, \tag{3}$$

where F is a distribution function and α is a positive rational number. However the distribution is well defined for any positive real α. Properties for a particular case of this distribution (with $F=\Phi$, the distribution function of the normal distribution) were studied in Gupta and Gupta (2008).

Durrans (1992), in a hydrological context, introduced the fractional order statistics distribution (that we call alpha-power distribution) with density function given by $\varphi_f(z;\alpha)=\alpha f(z)\{F(z)\}^{\alpha-1}$, $z\in\mathbb{R}$, $\alpha\in\mathbb{R}^+$, where F is the distribution function of the density f, which is symmetric about zero. We use the notation $Z\sim AP_f(\alpha)$. We refer to this model as the standard alpha-power distribution (see also Pewsey et al., 2012). This is an alternative asymmetric model which exhibits a broader range of asymmetry and kurtosis measures than does the skew-normal distribution (Azzalini, 1985). The parameter α is a shape parameter that controls the amount of asymmetry in the distribution.

Gupta and Gupta (2008) define the class of power-normal distributions whose distribution function is given by $[\Phi(z)]^{\alpha},\alpha>0$. That is, Z is said to follow a power-normal distribution (denoted $PN(\alpha)$) if its density function is given by $\varphi(z;\alpha)=\alpha\phi(z)\{\Phi(z)\}^{\alpha-1}$. If Z is a random variable from a standard $AP_f(\alpha)$ distribution then the location-scale extension of Z, $X=\xi+\eta Z$, where $\xi\in R$ and $\eta\in R^+$, has probability density function given by

$$\varphi_F(x;\xi,\eta,\alpha)=\frac{\alpha}{\eta}f\left(\frac{x-\xi}{\eta}\right)\left\{F\left(\frac{x-\xi}{\eta}\right)\right\}^{\alpha-1}. \tag{4}$$

We will denote this extension by using the notation $X\sim AP_f(\xi,\eta,\alpha)$.

As can be seen from Fig. 1, the parameter α controls both asymmetry and kurtosis. Moreover, it can be noticed that for $\alpha>1$, the kurtosis is greater than that of the normal distribution, and for $0<\alpha<1$ the opposite is observed.

Pewsey et al. (2012) derived the Fisher information matrix for the location-scale version of the power-normal model and have shown that it is nonsingular. Therefore, normality can be tested using ordinary large sample properties of the likelihood ratio statistics.

3. The bivariate alpha-power model

For the construction of bivariate alpha-power model, we will make use of the approach discussed in Arnold at al. (1992, 1999, 2001) based on conditional distributions. The bivariate random vector (X,Y) is said to be conditionally specified if for any possible value y of the random variable Y, the conditional distribution of X given $Y=y$ is a member of some known parametric family of distributions, and for any possible value x, the conditional distribution of Y given $X=x$ is a member of a second possibly but not necessarily different parametric family of distributions.

Consider now that $\{h_1(x;\underline{\omega}):\underline{\omega}\in\Omega\}$ denotes an exponential family of densities with l_1 parameters in $\mathbb{R}$ with respect to a measure μ_1 where $\Omega\in\mathbb{R}^{l_1}$. In addition, consider a possibly different exponential family of densities with l_2 parameters $\{h_2(y;\underline{\tau}):\underline{\tau}\in T\}$, with respect to some measure μ_2 where $T\in\mathbb{R}^{l_2}$.

Suppose, moreover, that for all $y\in S(Y)$ and for all $x\in S(X)$ (with $S(X)$ and $S(Y)$ being the value spaces for X and Y) we have that

$$f_{X|Y}(x|y)=h_1(x;\underline{\omega}(y)) \tag{5}$$