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On asymptotic normality of sequential estimators for branching processes with immigration

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Abstract

Consider a Galton–Watson process with immigration. This paper studies the limits of the sequential estimator, proposed by [Sriram, T.N., Basawa, I.V., and Huggins, R.M., (1991). Sequential estimation for branching processes with immigration. Ann. Statist. 19, 2232–2243.] and the modified sequential estimator, proposed by [Shete, S., Sriram, T.N., 1998. Fixed precision estimator of the offspring mean in branching processes. Stochastic Process. Appl. 77, 17–33.]. [Sriram, T.N., Basawa, I.V., Huggins, R.M., 1991. Sequential estimation for branching processes with immigration. Ann. Statist. 19, 2232–2243.] proved that the sequential estimators are asymptotically normal in the subcritical and critical cases. In this paper it is proved that the sequential estimators are asymptotically normal in the supercritical case and that the limiting distributions of the modified estimators, after being properly standardized, are normal as well.

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Keywords: Sequential estimator; Modified sequential estimator; Branching process; Stopping time; Asymptotic normality

1. Introduction

The study of Galton–Watson processes with immigration has drawn much attention since the work by Heyde and Seneta (1972, 1974), and early study on the estimation of means (m) of offspring numbers and immigration rates dates back to Smoluchowski (1916). Based only on the generation sizes, the maximum likelihood approach in the parametric models is, in general, too complicated to be useful, as noted by Heyde and Seneta (1972). However, if the numbers of immigrants are also observed, the approach yields very useful results under some parametric models. See, e.g., Bhat and Adke (1981), Venkataraman (1982) and Venkataraman and Nanthi (1982).

Many estimators of the offspring mean have been proposed and studied in Heyde and Seneta (1971), Heyde (1970), Quine (1976), Klimko and Nelson (1978), Seneta (1970) and Wei and Winnicki (1987) under subcritical ($m < 1$), critical ($m = 1$) or supercritical ($m > 1$) cases.

In an attempt to provide a unified estimator, Wei and Winnicki (1990) proposed the conditional weighted least squares method. It turns out that the limiting distribution for the estimator in the critical case, drastically different from that of the other two cases, is nonnormal.

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Later, Sriram et al. (1991) proposed a sequential estimator based on full information on both the generation sizes and the immigration process. It is shown that this sequential estimator is consistent and asymptotically normal in the subcritical case and the critical case. Recently, Shete and Sriram (1998) modified the sequential estimator and constructed a so-called fixed-precision estimator and proved that the modified estimator is unbiased in general and is asymptotically efficient as sequential estimator in the subcritical case and in the critical case. But some interesting questions remain unsolved. Whether or not the sequential estimator and the modified sequential estimator are asymptotically normal in the supercritical case are still open problems.

Recently, Qi and Reeves (2002) proposed a class of two-stage sequential estimators. These two-stage estimators are asymptotically normal in all three cases, and therefore, they can be used to construct the confidence interval for the mean parameter m without requirement of any prior information about m.

In the present paper we will study the limiting distributions of sequential estimators and modified sequential estimators. The contribution of the paper is to prove the asymptotic normality for the sequential estimators in Sriram et al. (1991) and the modified sequential estimators in Shete and Sriram (1998) in the supercritical case. It is worth mentioning that the modified sequential estimators seem not asymptotically normal in the supercritical case in Shete and Sriram (1998) as are shown in the simulation study of that paper. However, we will use some different random normalization constants for the modified sequential estimators in the present paper. That gives a unified normal limit for the modified sequential estimator in all three cases. Even so, we are unable to show in this paper that both the sequential and modified sequential estimators are uniformly normal in the supercritical case when the m is limited to a bounded interval. That might be true as suggested by the result of Shiryaev and Spokoiny (1997) for AR(1) processes when a standard normal error is assumed.

The paper is organized as follows. In Section 2, we introduce the sequential estimation methods, including the sequential estimators by Sriram et al. (1991), the modified sequential estimators by Shete and Sriram (1998) and the two-stage sequential estimators by Qi and Reeves (2002). In Section 3 we show that the sequential estimators are asymptotically normal in the supercritical case. In Section 4 we focus on study of the limiting distributions for the modified sequential estimators and prove that these estimators always have normal limits if (random) normalization constants are appropriately chosen.

2. Sequential estimation

Define the following branching process with immigration:

$$Z_n = \sum_{k=1}^{Z_{n-1}} \xi_{n-1,k} + Y_n, \quad n \geqslant 1, \tag{2.1}$$

where Z_n denotes the size of the nth generation of a population, Y_n the number of immigrants in the nth generation, and $\xi_{n-1,k}$ the number of offspring of the kth individual belonging to the $(n-1)$th generation. The initial population size Z_0 is a random variable independent of $\{\xi_{n,j}\}$ and $\{Y_n\}$. $\{\xi_{n-1,k}, n \geqslant 1, k \geqslant 1\}$ and $\{Y_n, n \geqslant 1\}$ are an independent array and sequence of independent and identically distributed (i.i.d) integer-valued random variables with unknown means m and λ, and variances $\sigma^2 \in (0, \infty)$ and $\sigma_Y^2 \in (0, \infty)$, respectively.

Throughout this paper we assume that both $\{Z_n\}$ and $\{Y_n\}$ are observed. An estimator for the offspring mean is given by

$$\widehat{m}_n = \sum_{i=1}^{n} (Z_i - Y_i) \Big/ \sum_{i=1}^{n} Z_{i-1}. \tag{2.2}$$

In order to achieve some better asymptotic properties for the estimation, Sriram et al. (1991) defined the stopping rule by

$$N_c = \inf\{n \geqslant 1 : \sum_{i=1}^{n} Z_{i-1} \geqslant c\sigma^2\}, \tag{2.3}$$

where $c > 0$ is some constant chosen appropriately. The sequential estimator of m is then given by $\widehat{m}_{N_c}$. They put the variance σ^2 in the definition of N_c in their paper in order to be able to prove the uniformly asymptotic normality and