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Generalized wordlength patterns and strength

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ABSTRACT

Xu and Wu (2001) defined the *generalized wordlength pattern* $(A_1,\ldots,A_k)$ of an arbitrary fractional factorial design (or orthogonal array) on k factors. They gave a coding-theoretic proof of the property that the design has strength t if and only if $A_1=\cdots=A_t=0$. The quantities A_i are defined in terms of characters of cyclic groups, and so one might seek a direct character-theoretic proof of this result. We give such a proof, in which the specific group structure (such as cyclicity) plays essentially no role. Nonabelian groups can be used if the counting function of the design satisfies one assumption, as illustrated by a couple of examples.

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1. Introduction

A *fractional factorial design* is a multisubset D of a finite Cartesian product $G=G_1\times\cdots\times G_k$, that is, a set of elements of G, the element $\mathbf{x}$ possibly repeated with some multiplicity $O(\mathbf{x})$. We will say that D is *based on* G, and refer to O as the *counting* or *multiplicity function*[1] of D. In statistical terminology, the set G_i indexes the levels of the ith factor in an experiment, and G is the set of treatment combinations. The treatment combinations used in the design are referred to as *runs*, and the number of runs in the design, counting multiplicities, is

$$|D|=\sum_{\mathbf{x}\in G}O(\mathbf{x}). \tag{1}$$

Xu and Wu (2001) associated to a design D a k-tuple $(A_1(D),\ldots,A_k(D))$, called its *generalized wordlength pattern*, defined as follows. If G_i has s_i elements, we take $G_i=\mathbb{Z}_{s_i}$, the additive cyclic group of integers modulo s_i. This makes G an abelian group. To each $u\in\mathbb{Z}_s$ we associate a complex-valued function χ_u on $\mathbb{Z}_s$ such that

$$\chi_u(x)=\xi^{ux}, \tag{2}$$

where ξ is a primitive sth root of unity (say $\xi=e^{2\pi i/s}$). For $\mathbf{u}=(u_1,\ldots,u_k)$ and $\mathbf{x}=(x_1,\ldots,x_k)\in G$, we let

$$\chi_{\mathbf{u}}(\mathbf{x})=\prod_i\chi_{u_i}(x_i), \tag{3}$$

and define the *J-characteristics*[2] of the design to be the quantities

$$\chi_{\mathbf{u}}(D)=\sum_{\mathbf{x}\in G}O(\mathbf{x})\overline{\chi_{\mathbf{u}}(\mathbf{x})}, \tag{4}$$

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[1] It is called the *indicator function* of D by a number of authors—for example, in Cheng and Ye (2004).

[2] When $s_1=\cdots=s_k=2$, the quantities $\chi_g(D)$ reduce to the *J-characteristics* of Deng and Tang (1999). We are following Ai and Zhang (2004) in using the same term for these quantities in the general case.

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the bar denoting the complex conjugate. This formula departs superficially from that given in Xu and Wu (2001). The introduction of the conjugate does not change the value of $\chi_{\mathbf{u}}(D)$ since the choice of ξ in (2) is arbitrary and may be replaced by $\xi = e^{-2\pi i/s}$. The factor $O(\mathbf{x})$ makes it explicit that each summand is repeated according to its multiplicity.

Finally, the "generalized wordlengths" are given by

$$A_j(D) = N^{-2} \sum_{\mathrm{wt}(\mathbf{u}) = j} |\chi_{\mathbf{u}}(D)|^2 \quad \text{for } j = 1, \ldots, k, \tag{5}$$

where $N = |D|$ is defined as in (1) and wt($\mathbf{u}$) is the *Hamming weight* of $\mathbf{u}$, that is, the number of non-zero components of $\mathbf{u}$. For the statistical meaning of the generalized wordlength pattern, the reader is referred to Xu and Wu (2001).

The design D may also be viewed as an orthogonal array, particularly if its runs are displayed in matrix form, say as columns of a $k \times N$ matrix. Xu and Wu (2001, Theorem 4(ii)) use a coding-theoretic result to show that $A_1(D) = \cdots = A_t(D) = 0$ iff D has strength t. They note in passing that the functions (2) and (3) are group characters, which might lead us to expect a character-theoretic proof of this result. Providing such a proof is the purpose of this paper.

Using a suggestive idea of Bierbrauer (1995), we first reexpress the numbers $A_j(D)$ in terms of certain Fourier coefficients.

The functions χ_u in (2) are the *irreducible characters* of the group $\mathbb{Z}_s$, and so the functions $\chi_{\mathbf{u}}$ are the irreducible characters of G. They form an orthonormal basis of the set of all functions from G to $\mathbb{C}$ under the inner product

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{\mathbf{x} \in G} \phi(\mathbf{x}) \overline{\psi(\mathbf{x})}. \tag{6}$$

If we express O in this basis as

$$O = \sum_{\mathbf{u} \in G} \mu_{\mathbf{u}} \chi_{\mathbf{u}},$$

then its Fourier coefficients $\mu_{\mathbf{u}}$ satisfy

$$\mu_{\mathbf{u}} = \langle O, \chi_{\mathbf{u}} \rangle = \frac{1}{|G|} \sum_{\mathbf{x} \in G} O(\mathbf{x}) \overline{\chi_{\mathbf{u}}(\mathbf{x})} = \frac{1}{|G|} \chi_{\mathbf{u}}(D),$$

so that the generalized wordlengths (5) are given by

$$A_j(D) = N^{-2} \sum_{\mathrm{wt}(\mathbf{u}) = j} |\chi_{\mathbf{u}}(D)|^2 = \frac{|G|^2}{N^2} \sum_{\mathrm{wt}(u) = j} |\mu_{\mathbf{u}}|^2.$$

To establish our claim, we need to show that D has strength t iff $\mu_{\mathbf{u}} = 0$ for all $\mathbf{u}$ such that $1 \le \mathrm{wt}(\mathbf{u}) \le t$.

It turns out that this result does not depend on the fact that the groups G_i are cyclic, or even abelian, although in the nonabelian case we will need to recast the concept of weight and to impose one restriction on O. We recast the main result in Section 4, and give the proof in Section 5. Background on character theory and on strength is given in Sections 2 and 3. We conclude with two examples illustrating the restriction on O in the nonabelian case.

There are many excellent expositions of character theory, and we will sometimes mention known results without citation. We will often refer to Isaacs (1976); other texts include Ledermann (1987) and Serre (1977).

Notation and terminology. As already indicated, the complex conjugate of z will be denoted by $\overline{z}$. We denote the complex numbers by $\mathbb{C}$, the integers modulo s by $\mathbb{Z}_s$, the cardinality of a set E by $|E|$, and vectors (k-tuples) by boldface. The set of complex-valued functions on G will be written as $\mathbb{C}^G$.

All groups are finite. The identity element of a group will generally be denoted by e.

When $G = G_1 \times \cdots \times G_k$ is a direct product of groups, the *Hamming weight* wt($\mathbf{u}$) of an element $\mathbf{u} \in G$ will be defined as the number of nonidentity components of $\mathbf{u}$. Here we have modified the usual definition of Hamming weight as G_i may have no zero symbol. Each G_i may be identified with a subgroup of G, namely the subgroup $e_1 \times \cdots \times e_{i-1} \times G_i \times e_{i+1} \times \cdots \times e_k$ where e_j is the identity of G_j. A similar identification holds for $G_{i_1} \times \cdots \times G_{i_m}$ where $1 \le i_1 < \cdots < i_m \le k$. For such subgroups it will be useful to introduce the following terminology.

Definition 1.1. If $H = G_{i_1} \times \cdots \times G_{i_m}$, we call H a *factorial subgroup* of G. The number m will be called the *rank* of H. The *factorial complement* of H in G is $\prod_{i \notin I} G_i$, where $I = \{i_1, \ldots, i_m\}$.

2. Characters

We will deal only with complex-valued characters. We refer the reader to a treatment of character theory for more detail, and simply quote the results that we will need.

The set of characters on the group G is closed under pointwise addition, and contains a finite set Irr(G) that generates it in the sense that every character on G is a unique linear combination of characters in Irr(G) with nonnegative integer coefficients. The characters in Irr(G) are called *irreducible*. Among them is the *principal character* $\chi \equiv 1$. The irreducible characters of the cyclic group $\mathbb{Z}_s$ are given by (2), while for an abelian group G they are the homomorphisms from G to the multiplicative group $\mathbb{C}^*$ (Isaacs, 1976, Corollary 2.6).

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