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Spatial process gradients and their use in sensitivity analysis for environmental processes

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ABSTRACT

This paper develops methodology for local sensitivity analysis based on directional derivatives associated with spatial processes. Formal gradient analysis for spatial processes was elaborated in previous papers, focusing on distribution theory for directional derivatives associated with a response variable assumed to follow a Gaussian process model. In the current work, these ideas are extended to additionally accommodate a continuous covariate whose directional derivatives are also of interest and to relate the behavior of the directional derivatives of the response surface to those of the covariate surface. It is of interest to assess whether, in some sense, the gradients of the response follow those of the explanatory variable. The joint Gaussian structure of all variables, including the directional derivatives, allows for explicit distribution theory and, hence, kriging across the spatial region using multivariate normal theory. Working within a Bayesian hierarchical modeling framework, posterior samples enable all gradient analysis to occur post model fitting. As a proof of concept, we show how our methodology can be applied to a standard geostatistical modeling setting using a simulation example. For a real data illustration, we work with point pattern data, deferring our gradient analysis to the intensity surface, adopting a log-Gaussian Cox process model. In particular, we relate elevation data to point patterns associated with several tree species in Duke Forest.

1. Introduction

Increasingly, data is being collected at geo-referenced locations. For a region of interest D, the set of conceptual responses $\{Y(\mathbf{s}) : \mathbf{s} \in D\}$ can be viewed as a realization of a random surface, observed at a finite set of locations. While covariate information may explain a substantial portion of the variation in response, there is often underlying spatial structure that is difficult to measure. Inference on this spatial structure can be made via the parameters in a spatial process model. Under these models, prediction at unobserved locations, or kriging, is possible, enabling interpolation across the region.

Spatial regression models commonly assume a linear relationship and make inference based on the coefficient assigned to the covariate. This coefficient describes the expected change in response given a unit change in covariate, thus providing a global measure for the sensitivity of the response to the covariate. However, it is expected that the relationship between

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the variables may vary locally over the study region. Such local, or second-order, behavior can be studied through spatial sensitivity or gradient analysis.

A spatial gradient analysis will enable spatial examination of a response variable's sensitivity to a covariate across the region of interest. The sensitivity of the response variable may vary based on the rate of change for the covariate or due to additional unaccounted for factors, resulting in areas where the relationship appears weaker or stronger. Models allowing for spatially varying coefficients, such as Fotheringham et al. (2002) and Gelfand et al. (2003), provide some similar inference but assume a more complex model structure. The methodology proposed here assumes a standard spatial linear regression model but provides a post model fitting framework for examining the variation in the response's sensitivity to the covariate.

In ecology such sensitivities are typically discussed when relating plant characteristics to climate. For instance, researchers are increasingly interested in characterizing how abundance and frequency of tree species relate to changes in variables such as temperature and precipitation, in order to learn about the expected effects of climate change on range distributions (e.g. Thuiller et al., 2004; Canham and Thomas, 2010; Thomas, 2010). These analyses focus on a comparison of sensitivities across species, while little consideration is given to how sensitivities may vary spatially within a given species. Our approach is aimed at the latter question.

The interpretation of the coefficient in a spatial regression as a global gradient, $dE(Y(\mathbf{s}))/dX(\mathbf{s})$, inspires consideration of local sensitivities through directional derivative processes, or spatial gradients. Spatial gradients under Gaussian processes were elaborated by Banerjee et al. (2003b) to address the rate of change of a spatial surface at a given point in a given direction. Their paper defines directional derivative processes with corresponding distribution theory to enable interpolation across a region. The gradient distributions are fully determined by the spatial model parameters, allowing all gradient analysis to occur post model fitting. Distributions for derivatives of Gaussian processes have also been discussed in the context of observed derivatives of functions (e.g. O'Hagan, 1992; Solak et al., 2003), as well as for random fields more generally (Adler, 1981). In all of the previous work with Gaussian process directional derivatives (e.g. Banerjee and Gelfand, 2003, 2006; Majumdar et al., 2006), the researchers have considered the rates of change of a response surface with the mean surface modeled as a linear function of a set of fixed covariates. In contrast, to accommodate the desired spatial sensitivity analysis, we will assume a single covariate of interest whose surface is spatially smooth such that it too can be treated as a realization of a stochastic process. The behavior of the response and covariate processes, as well as their associated derivative processes, are then considered jointly and functions of these derivatives can be explored.

The contribution of this paper is to extend the existing spatial gradient theory to accommodate spatial sensitivity analysis by modeling the response and covariate jointly. Working within a hierarchical Bayesian modeling framework, corresponding gradients for the spatial surfaces can be sampled simultaneously from the joint predictive distribution post model fitting. Under a significant regression relationship it is not sensible to investigate the gradient behavior of the surfaces marginally. Suitable comparison between the gradient surfaces illustrates how sensitive the response surface is to the covariate surface, as well as the strength of this relationship. The former is accomplished through comparison between the directions of the maximum gradient at a given location; the latter requires consideration of their directional derivatives relative to one another. In particular, we introduce two new spatial processes, a local directional sensitivity process and a spatial angular discrepancy process. These inferential tools are developed and carried out on simulated data in the context of a customary geostatistical model (Banerjee et al., 2003a; Cressie and Wikle, 2011) as well as with an ecological data set where we connect point patterns of trees with elevation.

In Section 2 the formal distribution theory for the spatial gradients is extended to the multivariate case. Section 3 outlines the modeling framework for our examples. This section also defines the two processes of interest, namely the local directional sensitivity process and the spatial angular discrepancy process. Section 4 provides a simulated example with a multivariate Gaussian process setup as a proof of concept. Section 5 provides an analysis of point pattern data from Duke Forest, extending the analysis techniques to a non-Gaussian response; the intensity of the point pattern, modeled using a log-Gaussian Cox process, is explored by employing a spatial gradient chain rule. Finally, Section 6 summarizes the contributions of the paper and suggests future work.

2. Distribution development

In this section we review the definitions and distributions presented in Banerjee et al. (2003b) and extend these ideas to consider a multivariate Gaussian process. We assume locations $\mathbf{s} \in \mathbb{R}^2$, 2-dimensional Euclidean space, however extension to a generic d-dimensional setting is straightforward. The process is assumed, for convenience, to be (weakly) stationary such that the covariance function, $Cov(Y(\mathbf{s}), Y(\mathbf{s}'))$, depends only on the separation vector $\boldsymbol{\delta} = \mathbf{s} - \mathbf{s}'$. In fact, in our examples we adopt isotropic covariance functions that depend only on the length of the separation vector, $\|\boldsymbol{\delta}\|$.

Consider two surfaces $\{(Y(\mathbf{s}), X(\mathbf{s})) : \mathbf{s} \in \mathbb{R}^2\}$ drawn from a joint Gaussian process specified such that $X(\mathbf{s})$ has constant mean, say α_0, and covariance function $G(\boldsymbol{\delta})$. Given $X(\mathbf{s})$, $Y(\mathbf{s})$ has mean $\beta X(\mathbf{s})$ and covariance function $K(\boldsymbol{\delta})$. Observed at a set of locations $\mathbf{Y} = (Y(\mathbf{s}_1), \ldots, Y(\mathbf{s}_n))$, we write: $\mathbf{Y}|\mathbf{X} \sim N(\beta\mathbf{X}, K(\cdot))$ with $\mathbf{X} \sim N(\alpha_0, G(\cdot))$. Considered jointly, we have:

$$\begin{pmatrix}\mathbf{Y}\\ \mathbf{X}\end{pmatrix} \sim N\left(\begin{pmatrix}\alpha_0\beta\mathbf{1}\\ \alpha_0\mathbf{1}\end{pmatrix}, \begin{pmatrix}K(\cdot)+\beta^2G(\cdot) & \beta G(\cdot)\\ \beta G(\cdot) & G(\cdot)\end{pmatrix}\right)$$

where $G(\cdot)$ and $K(\cdot)$ are matrices of the covariance functions with entry i, j evaluated at $\boldsymbol{\delta} = \mathbf{s}_i - \mathbf{s}_j$.