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Transient probability functions of finite birth–death processes with catastrophes

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Abstract

A representation of the transient probability functions of finite birth–death processes (with or without catastrophes) as a linear combination of exponential functions is derived using a recursive, Cayley–Hamilton approach. This method of solution allows practitioners to solve for these transient probability functions by reducing the problem to three calculations: determining eigenvalues of the Q-matrix, raising the Q-matrix to an integer power and solving a system of linear equations. The approach avoids Laplace transforms and permits solution of a particular transition probability function from state i to j without determining all such functions.

MSC: 60K25; 60J27

Keywords: Finite birth–death processes; Catastrophe processes; Dual processes; Transient probability functions; Recursive; Cayley–Hamilton approach

1. Introduction

The transient probability functions of finite birth–death processes have been established in the literature for many years, see, for example Ledermann and Reuter (1954), Rosenlund (1978), Mohanty et al. (1993), Kijima (1997). The standard analysis to derive this result usually makes use of birth–death polynomials, spectral analysis, numerical analysis and Laplace transforms. In this article, a different approach is taken. After characterizing the eigenvalues of certain tridiagonal matrices, an explicit representation of the transient probability functions of finite birth–death processes is proved using a recursive, Cayley–Hamilton method. This technique evolved from the work of Green et al. (2003), although a matrix variation of this idea has been traced back to one of the ways to compute an exponential matrix discussed in the enlightening articles by Moler and van Loan (1978, 2003). This recursive, Cayley–Hamilton approach for solving finite birth–death processes is appealing from a pedagogical and practitioner's point of view because it reduces the problem of finding transient probability functions to three basic operations: determining eigenvalues of a matrix, raising a matrix to an integer power and solving a system of linear equations.

Today there is a resurgence of interest in analyzing Markov processes having catastrophe transitions, see Chang et al. (2005), Chao and Zheng (2003), Di Crescenzo et al. (2003), Krinik et al. (2005), Kumar and Arivudainambi (2000),

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Swift (2000). The recursive, Cayley–Hamilton approach is also applied in this article to obtain an explicit representation of the transient probability functions of finite birth–death processes having constant catastrophe transition rates. It turns out, surprisingly, that the general form of these transient probability functions remains the same for birth–death processes with catastrophes as for birth–death processes. This representation of the transient probability functions for birth–death processes with catastrophes appears new. In place of numerical analysis, the theory of dual processes is utilized to obtain these results. Once again, the explicit computation of these transient probability functions reduces to finding eigenvalues, taking powers of a matrix, and solving a system of linear equations.

Section 2 of this article is devoted to reviewing the necessary preliminaries on linear, constant coefficient recurrence relations, eigenvalues, the Cayley–Hamilton Theorem, Vandermonde matrices and theorems concerning the eigenvalues of certain, real tridiagonal matrices. In Section 3, the recursive, Cayley–Hamilton approach is presented to find a representation of the transient probability functions for finite birth–death processes as a linear combination of exponential functions. A similar representation formula for birth–death processes with catastrophes is deduced in Section 4. The key ideas here are the relationship between a Markov process and it's dual process. Examples of transient probability functions are calculated in Section 5 for both a finite birth–death process and a finite birth–death process with catastrophes. Section 6 is a short concluding section where advantages of the recursive, Cayley–Hamilton approach are summarized and related problems are discussed.

2. Preliminaries

2.1. Recurrence relations

Recurrence relations play an important role in this article.

Definition 2.1. Given a homogeneous linear recurrence relation with constant coefficients of order m,

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_m x_{n-m} \quad (a_m \neq 0).$$

Then the polynomial equation

$$x^m - a_1 x^{m-1} - a_2 x^{m-2} - \cdots - a_m = 0$$

will be called the characteristic equation of the above recurrence relation.

By the Fundamental Theorem of Algebra, this equation has m (possibly complex) roots, $z_1, z_2, \ldots, z_m$, called the characteristic roots. If the roots are distinct, then a general solution of the homogeneous linear recurrence relation can be written as

$$x_n = c_1 z_1^n + c_2 z_2^n + \cdots + c_m z_m^n = \sum_{l=1}^{m} c_l z_l^n.$$

This well-known result may be found for example in Jackson and Thoro (1990). The coefficients of this linear combination are constants determined by the initial conditions of the recurrence relation.

2.2. Matrices, eigenvalues, and the Cayley–Hamilton theorem

The necessary definitions and results are reviewed here for reference. Recall the definition of the characteristic polynomial of a matrix.

Definition 2.2. The characteristic polynomial of an $m \times m$ matrix Q is $f(x) = \det(Q - xI)$.

Notice that the characteristic polynomial is an mth degree polynomial that can be written as $f(x) = (-1)^m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$. Setting the characteristic polynomial equal to zero gives the characteristic equation of the matrix Q, whose roots are the eigenvalues of Q.