Journal of Statistical Planning and Inference 142 (2012) 2501–2517

Contents lists available at SciVerse ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Consistency and robustness of tests and estimators based on depth

Liesa Denecke*, Christine H. Müller

Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany

A R T I C L E I N F O

Article history:
Received 10 October 2011
Received in revised form
28 March 2012
Accepted 29 March 2012
Available online 8 April 2012

Keywords:
Breakdown point
Consistency
Data depth
Gaussian copula
Gumbel copula
Likelihood depth
Parametric estimation
Robustness
Simplicial depth
Tests
Weibull distribution

A B S T R A C T

In this paper it is shown that data depth does not only provide consistent and robust estimators but also consistent and robust tests. Thereby, consistency of a test means that the Type I (α) error and the Type II (β) error converge to zero with growing sample size in the interior of the nullhypothesis and the alternative, respectively. Robustness is measured by the breakdown point which depends here on a so-called concentration parameter. The consistency and robustness properties are shown for cases where the parameter of maximum depth is a biased estimator and has to be corrected. This bias is a disadvantage for estimation but an advantage for testing. It causes that the corresponding simplicial depth is not a degenerated U-statistic so that tests can be derived easily. However, the straightforward tests have a very poor power although they are asymptotic α-level tests. To improve the power, a new method is presented to modify these tests so that even consistency of the modified tests is achieved. Examples of two-dimensional copulas and the Weibull distribution show the applicability of the new method.

1. Introduction

Most approaches in robustness against outliers and contamination concern estimation. The absolute minority deals with robust testing, see e.g. the books on robust statistics of Jurecková and Sen (1996), Müller (1997), Maronna et al. (2006), Huber and Ronchetti (2009), and Heritier et al. (2009). Robust tests are typically asymptotic α level tests, since it is impossible to derive the finite sample distribution. Moreover, the behavior under the alternative is unknown. Even consistency is usually not proved for these robust tests, although this is an important property of a test. Thereby a test $\varphi_N : \mathcal{Z}^N \to \{0,1\}$ for $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta \backslash \Theta_0$ is called consistent, if

$$\lim_{N\to\infty} P_\theta(\varphi_N = 1) = 0 \quad \text{for all } \theta \in \text{int}(\Theta_0),$$

$$\lim_{N\to\infty} P_\theta(\varphi_N = 1) = 1 \quad \text{for all } \theta \in \text{int}(\Theta \backslash \Theta_0),$$

where $\text{int}(A)$ denotes the interior of a set A. Hence consistency of a test concerns the Type I as well as the Type II error. In this paper, we prove the consistency of some robust tests based on data depth. Since consistency and robustness of a test are strongly related to consistency and robustness of corresponding estimators, we also treat both for estimators based on data depth.

* Corresponding author. Tel.: +49 2317553114; fax: +49 2317553454.
E-mail address: denecke@statistik.tu-dortmund.de (L. Denecke).

0378-3758/$ - see front matter © 2012 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.jspi.2012.03.024

The concept of data depth is an approach to generalize the median and ranks to multivariate data and more complex situations. Notions of data depth were for example developed for multivariate data by Tukey (1975), Liu (1990) and Mosler (2002). Meanwhile data depth is a wide spread method with many applications, see e.g. Lin and Chen (2006), Li and Liu (2008), Romanazzi (2009), López-Pintado and Romo (2009), López-Pintado et al. (2010), Hu et al. (2011) for some recent applications. Zuo and Serfling (2000a,b) provided some general properties of depth notions. An important step for more generalizations was the development of the concept of the nonfit. Depth notions defined via the nonfit were at first developed by Rousseeuw and Hubert (1999) for regression. Mizera (2002) extended this approach by using general quality functions for the nonfit and by distinguishing between global and tangent depth. Quality functions can be obtained in particular by likelihood functions. This leads to the concept of likelihood depth which can be applied to general situations. For example, it was applied to location and scale depth in Mizera and Müller (2004). Estimators maximizing the likelihood depth are robust alternatives to the nonrobust maximum likelihood estimators (MLE). The estimators based on likelihood depth are comparable concerning flexibility and applicability to the MLE. The latter is depending very much on the underlying distribution, small changes in the distribution can lead to extreme adulterations in the estimator. This is not the case for the estimators maximizing the likelihood depth. In particular, in simulation studies these estimators showed high robustness against contaminations with other distributions, but the robustness was not proved formally yet. See e.g. Wellmann et al. (2007) and Denecke and Müller (2011).

Any depth notion can be used to define simplicial depth as Liu (1990) did using the halfspace depth of Tukey (1975). Every simplicial depth is an U-statistic so that its asymptotic distribution is in principle known and asymptotic α-level tests can be derived, see Müller (2005). In particular, general hypotheses of the form $H_0 : \theta \in \Theta_0$ can be tested by using the maximum simplicial depth over the set Θ_0. However, many simplicial depth notions are degenerated U-statistics so that the spectral decomposition of a conditional expectation is needed to derive the asymptotic distribution which in this case is not a normal distribution. Such spectral decompositions were for example derived for polynomial regression, multiple regression, and orthogonal regression by Wellmann et al. (2009) and Wellmann and Müller (2010a,b), respectively. There are also cases where the simplicial depth is not a degenerated U-statistic, for example the simplicial depth based on likelihood depth for copulas, as Denecke and Müller (2011) showed. Nondegenerated U-statistics have the normal distribution as asymptotic distribution so that asymptotic α-level tests can be derived easily. However, the price for a nondegenerated simplicial is a bias of the estimator maximizing the underlying depth notion. This bias caused also a very bad asymptotic power of the tests for some alternatives in simulation studies (see Denecke and Müller, 2011).

In this paper, we provide a robustness proof and an explanation for the empirically observed bad power at some alternatives. In particular, we prove that the straightforward tests are not consistent but that they can be corrected so that they are consistent. Further, we show that the correction is different from the correction which is needed to obtain the consistency of the corresponding maximum depth estimators. But they are related as well as the robustness properties. Therefore we also treat the consistency and robustness of estimators based on maximum depth. The robustness is derived via the breakdown point. Since we also allow bounded parameter space, it is shown that the breakdown point depends on a so-called concentration parameter which counts the number of observations in special subsets of the observation space. These subsets can be empty sets, single points, linear subspaces or larger subsets.

The consistency and robustness of estimators and tests are shown for depth based on general quality functions and one-dimensional parameters. We think that this approach also can be used for multidimensional parameters, but this needs much more technical effort. In particular, it is more difficult to derive examples. Therefore we only sketch the possible extension for multidimensional parameters. The examples considered here are derived via likelihood depth. These examples concern univariate data coming from a Weibull distribution with known shape or known scale and bivariate data given by the Gumbel or Gaussian copula. The concentration parameter of these examples are based on subsets which are a empty set (scale parameter of the Weibull distribution), a single point (shape parameter of the Weibull distribution), linear subspaces (Gaussian copula) and also a larger subset (Gumbel copula).

The paper is organized as follows. Section 2 concerns the consistency and the breakdown point of estimators based on data depth. It presents the new characterization of the breakdown point via the concentration parameter. This result together with the consistency of the estimators provides the notations and foundations which are needed to derive the consistency and robustness of the tests in Section 3. In particular, the results for the examples are needed to derive later consistent and robust tests for these examples. The completely new approach for testing is presented in Section 3. It provides consistency and robustness of tests based on depth for hypotheses given by $H_0 : \theta \le \theta_0$, $H_0 : \theta \ge \theta_0$, and $H_0 : \theta = \theta_0$. Thereby, also the inconsistency of the straightforward tests is proved. Each section ends by treating the examples of Weibull distribution and the copulas. Before the examples are given, a sketch of the possible extension for multidimensional parameters is given in each of Sections 2 and 3. All proofs can be found in the Appendix.

2. Estimators based on depth

2.1. Depth and maximum depth estimators

Let $Z_1, \ldots, Z_N$ be i.i.d. random variables with values in $\mathcal{Z} \subset \mathbb{R}^p$ and with continuous distribution P_θ and density $f_\theta, \theta \in \Theta \subset \mathbb{R}^q$. As in Mizera (2002) let $Q : \Theta \times \mathcal{Z} \to \mathbb{R}$ denote a quality function that measures for every data z_n, $n = 1, \ldots, N$,