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A limit theorem of *D*-optimal designs for weighted polynomial regression

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A B S T R A C T

Consider the *D*-optimal designs for the *d*th-degree polynomial regression model with a continuous weight function on a compact interval. As the degree of the model goes to infinity, we derive the asymptotic value of the logarithm of the determinant of the *D*-optimal design. If the weight function is equal to 1, we derive the formulae of the values of the *D*-criterion for five classes of designs including (i) uniform density design; (ii) arcsin density design; (iii) $J_{1/2,1/2}$ density design; (iv) arcsin support design and (v) uniform support design. The comparison of *D*-efficiencies among these designs is investigated; besides, the asymptotic expansions and limits of their *D*-efficiencies are also given. It shows that the *D*-efficiency of the arcsin support design is the highest among the first four designs.

1. Introduction

Consider the weighted polynomial regression model of degree $d \geq 1$,

$$y(x) = \sum_{i=0}^{d} \beta_i x^i + \varepsilon(x),$$
$$\mathrm{Var}(y(x)) = \sigma^2/\omega(x), \tag{1.1}$$

where $\varepsilon(x)$ is a random error component. Let $\beta = (\beta_0, \beta_1, \ldots, \beta_d)^T$ denote the vector of unknown parameters and $f_d(x) = (1, x, \ldots, x^d)^T$ is the vector of regression functions. The errors are assumed to have mean zero and unknown variance $\sigma^2/\omega(x)$. Suppose that $\sigma^2 > 0$ is a fixed unknown parameter, $\omega(x)$ denotes a continuous weight function on a compact design interval $I = [a, b] \subseteq \mathbb{R}$ and the control variable x is taken from I. Additionally, we usually assume that the errors are uncorrelated.

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The model (1.1) is widely used in situations where the response is curvilinear, because complex nonlinear relationships can be adequately modeled by polynomials over reasonably small range of the x's, and the variance of an observation depends on the explanatory variable in the hypothesized model, as in the case with some econometric models. For example, if the response variable is household expenditure and one explanatory variable is household income; then the variance of the observations may be a function of household income.

An approximate design ξ_d is an arbitrary probability measure on the design interval I. The information matrix of a design ξ_d for the parameter β is defined by

$$M(\xi_d)=\int_a^b \omega(x) f_d(x) f_d^T(x)\,\mathrm{d}\xi_d(x).$$

A design ξ_d^* is approximate D-optimal for β if ξ_d^* maximizes the determinant of the information matrix $M(\xi_d)$ among the set of all approximate designs on I. It is the same as minimizing the volume of the confidence ellipsoid for β if $\varepsilon(x)$ is normally distributed. For more about the theory of optimal designs see Fedorov (1972), Silvey (1980), and Pukelsheim (1993).

Hoel (1958) proved that for the model (1.1) on $[-1,1]$ with $\omega(x)=1$, the D-optimal design is concentrated with equal weights at the zeros of $(1-x^2)P_d'(x)$, where $P_d'(x)$ is the derivative of the dth-degree Legendre polynomial $P_d(x)$. Theorem 2.3.3 of Fedorov (1972) provided a differential equation approach to determine the D-optimal designs for a weighted polynomial regression model. This approach was used in Karlin and Studden (1966), Huang et al. (1995), Chang and Lin (1997), Imhof et al. (1998), Dette et al. (1999), and Antille et al. (2003), among others.

Fedorov (1972, p. 91) and Kiefer and Studden (1976) showed that the D-optimal design for the model (1.1) on $[-1,1]$ with $\omega(x)=1$ converges weakly to the arcsin distribution on $[-1,1]$ as d tends to infinity. Dette and Wong (1995) and Dette (1997) also investigated the limiting distribution of D-optimal designs. Chang (1998) proved that the D-optimal designs for model (1.1) on $[a,1]$, $0\le a<1$ with $\omega(x)=x^{2s+2}$, $s>-1$ converge weakly to the arcsin distribution on $[a,1]$. Chang et al. (2009) showed that the D-optimal design for the model (1.1) with $\omega(x)=\exp(\alpha x)$, $\alpha\in\mathbb{R}$, also converges weakly to the arcsin distribution on $[a,b]$ as d tends to infinity. Chang and Lin (2008) proved that the minimally-supported D-optimal design for the model (1.1) also converges weakly to the arcsin distribution on $[a,b]$ as d tends to infinity. Dette and Studden (1992) derived a new characterization of the classical orthogonal polynomials. Dette and Studden (1995) investigated the asymptotic properties for the zeros of the Jacobi, Laguerre and Hermite polynomials.

The main aim of this paper is to determine the asymptotic value of $\ln\det M(\xi_d^*)$ as $d\to\infty$. The main tools are the Euler–Maclaurin summation formula and the optimal value of the D-criterion for the model (1.1) on $[-1,1]$ with $\omega(x)=1$ (see Schoenberg, 1959).

The paper is organized in the following way. In Section 2, we first state a formula (see Schoenberg, 1959) about the optimal value of the D-criterion for the model (1.1) on $[-1,1]$ with $\omega(x)=1$. Then the asymptotic expansion of the optimal value is determined by the Euler–Maclaurin summation formula. In Section 3, we show that the ratio of the log of the maximal determinants of the optimal designs with an arbitrary continuous weight function and the constant weight function on $[-1,1]$ goes to 1 as $d\to\infty$ with the help of lemmas in Section 2. In Section 4, if the weight function is equal to 1, we derive the formulae of the values of the D-criterion for five classes of designs including (i) uniform density design; (ii) arcsin density design; (iii) $J_{1/2,1/2}$ design; (iv) arcsin support design and (v) uniform support design. The comparison of D-efficiencies among these designs is investigated; besides, the asymptotic expansions and limits of their D-efficiencies are also given. In Section 5, we make a description of some concluding remarks. Finally, all the proofs of lemmas are deferred to the Appendix.

2. Preliminaries

Consider any approximate design ξ_d for the model (1.1) on $[-1,1]$ with $\omega(x)=1$. The following lemma gives the optimal value of the D-criterion of ξ_d by Schoenberg (1959). Moreover, it shows that how to choose the support points and the corresponding weights of ξ_d can attain the optimal value.

Lemma 2.1 (*Theorem 5 of Schoenberg, 1959, p. 284*). *Let ξ_d be a design for the model (1.1) on $[-1,1]$ with $\omega(x)=1$. Then*

$$\det M(\xi_d)\le 2^{d(d+1)}\left(\prod_{k=1}^{d}k^k\right)^4 d^{-d}\prod_{k=1}^{2d}k^{-k},$$

with equality if and only if ξ_d assigns equal weights $1/(d+1)$ to the $d+1$ points x_i that solve the equation

$$(1-x^2)P_d'(x)=0,$$

where $P_d'(x)$ is the derivative of the dth-degree Legendre polynomial $P_d(x)$.

The Euler–Maclaurin summation formula is the shortest and most efficient way to deal with $\sum_{a\le k<b}g(k)$. The following lemma is an application of this formula to find the approximate expression and the remainder term of $\sum_{k=1}^{n}(k+a)\ln(k+a)$.