Journal of Statistical Planning and Inference 154 (2014) 72–86

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Operator-valued spectral measures and large deviations

Fabrice Gamboa [a,*], Alain Rouault [b]

[a] Université P. Sabatier, Institut de Mathématiques de Toulouse, 118 route de Narbonne, F-31062 Toulouse Cedex 9, France
[b] Université Versailles-Saint-Quentin, LMV UMR 8100, 45 Avenue des Etats-Unis, F-78035 Versailles Cedex, France

ARTICLE INFO

Available online 18 February 2014

Keywords:
Random matrices
Spectral measures
Matrix orthogonal polynomials
Large deviations

ABSTRACT

Let $\mathfrak{H}$ be a Hilbert space, let U be a unitary operator on $\mathfrak{H}$ and let $\mathfrak{K}$ be a cyclic subspace for U. The spectral measure of the pair $(U, \mathfrak{K})$ is an operator-valued measure $\mu_{\mathfrak{K}}$ on the unit circle $\mathbb{T}$ such that

$$\int_{\mathbb{T}} z^k \, d\mu_{\mathfrak{K}}(z) = (P_{\mathfrak{K}} U^k)_{\restriction \mathfrak{K}}, \quad \forall k \geq 0$$

where $P_{\mathfrak{K}}$ and $\restriction \mathfrak{K}$ are the projection and restriction on $\mathfrak{K}$, respectively. When $\mathfrak{K}$ is one dimensional, μ is a scalar probability measure. In this case, if U is picked at random from the unitary group $\mathbb{U}(N)$ under the Haar measure, then any fixed $\mathfrak{K}$ is almost surely cyclic for U. Let $\mu^{(N)}$ be the random spectral (scalar) measure of $(U, \mathfrak{K})$. The sequence $(\mu^{(N)})$ was studied extensively, in the regime of large N. It converges to the Haar measure λ on $\mathbb{T}$ and satisfies the Large Deviation Principle at scale N with a good rate function which is the reverse Kullback information with respect to λ (Gamboa and Rouault, 2010). The purpose of the present paper is to give an extension of this result for general $\mathfrak{K}$ (of fixed finite dimension p) and eventually to offer a projective statement (all p simultaneously), at the level of operator-valued spectral measures in infinite dimensional spaces.

1. Introduction

1.1. *The influence of the mathematical work of Studden on our research*

A significant part of the mathematical contribution of W.J. Studden relies on moment problems or more generally on generalized moment problems for *T-systems*. The first author of the present paper first met the *T-systems* during his Ph.D. preparation by the fascinating reading of two books on moment problems. The first one is the book of Krein and Nudel'man (1977) dealing mainly with the Markov moment problem. The second one is the book of Karlin and Studden (1966) that offers a beautiful journey inside the continent of *T-systems* properties. The reading of these two books has whetted our interest for the literature on moment problems and by the middle of the nineties we came across a very interesting paper of Chang et al. (1993) on the asymptotic behavior of randomized moment sequences. This seminal paper gives a very nice Borel Poincaré like theorem for moment sequences of probability measures on the unit interval and has been quite motivating for at least the last ten years of our researches. The probabilized moment space frame developed therein led to many papers written by many authors (see for example Gupta, 2000; Gamboa and Lozada-Chang, 2004; Lozada-Chang, 2005; Dette and Gamboa, 2007; Dette and Nagel, 2012; Gamboa et al., 2012). One of the main ingredient tools for the study of probabilized moment spaces is the parametrization of these spaces by

* Corresponding author.
E-mail addresses: fabrice.gamboa@math.univ-toulouse.fr (F. Gamboa), alain.rouault@uvsq.fr (A. Rouault).

http://dx.doi.org/10.1016/j.jspi.2014.02.005
0378-3758

the canonical moments. Roughly speaking, under natural probability measures these parameters have a joint product law with beta marginal distributions. They are also very intriguing nice mathematical objects with a lot of properties that we learned from the excellent book of Dette and Studden (1997). Moreover, studying the exhaustive books of Simon (2005), we realized that canonical moments, also called Verblunsky coefficients, are quite important objects in complex analysis and spectral theory. At this time the second author of the present paper was working on the asymptotic properties of the determinant of classical random matrix ensembles (Rouault, 2007). Surprisingly, by the Bartlett formula, the distribution of these random determinants involves product of independent beta random variables having similar parameters to those found in the randomized moment problem. Observing this analogy, we discovered a connection between random moments and spectral measures of classical random matrix ensembles (Gamboa and Rouault, 2010, 2011). The present paper is a matricial extension of the asymptotic studies conducted in the latter papers dealing with scalar random spectral measures. This extension has been possible thanks to two significant contributions of Dette and Studden in the field of matricial moment problems (see Dette and Studden, 2002, 2003).

We never had the opportunity to meet W.J. Studden but we wish to pay here a tribute to this creative mathematician that had often enlightened the paths of our researches.

1.2. Introduction to this paper

To capture the asymptotic behavior of large dimensional unitary random matrices, the usual statistic is the empirical spectral distribution, providing equal weights to all eigenvalues

$$\mu_{\mathrm{u}} = \frac{1}{N}\sum_{k=1}^{N} \delta_{\lambda_k}.$$

More recently, some authors used another random probability measure based on eigenvalues and eigenvectors (Killip and Nenciu, 2004; Bai et al., 2007; Gamboa and Rouault, 2010). If U is a unitary matrix and e is a fixed vector (assumed to be cyclic), the so-called spectral measure $\mu_{\mathrm{w},1}$ of the pair (U,e) may be defined through its algebraic moments. Indeed, for all $n \in \mathbb{Z}$

$$\langle e, U^n e \rangle = \int_{\mathbb{T}} z^n \, d\mu_{\mathrm{w},1}(z).$$

Here, $\mathbb{T}$ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. The measure $\mu_{\mathrm{w},1}$ is finitely supported on the eigenvalues of U, we may write

$$\mu_{\mathrm{w},1} = \sum_{k=1}^{N} \mathrm{w}_k \delta_{\lambda_k}$$

where w_k is the square of the scalar product of e with a unitary eigenvector associated with λ_k. The latter measure carries more information than the former. The weights in $\mu_{\mathrm{w},1}$ are blurred footprints of the eigenvectors of U. To make these footprints unblurred, it is then tempting to try to increase the *dimension* by *projecting* U on a fixed subspace of dimension p instead on the span generated by the single vector e. We obtain a matrix measure. This is what we will do in this paper. Actually we may even go back to the representation given by the spectral theorem (see Dunford and Schwartz, 1988)

$$U = \int_{\mathbb{T}} z E_U(dz)$$

where E_U is the spectral measure of U (or resolution of the identity for U). In our work, we sample U according to the Haar distribution on $\mathbb{U}(N)$ and we study the random object E_U.

The paper is organized as follows. In the next section, we first frame our paper by giving the main notations and definitions needed further. Then, we recall some facts on unitary matrices and matrix orthogonal polynomials on the unit circle. We also show technical results on these objects that will be useful later. In Section 3 we first study the effects of the randomization of the unitary matrices on the object defined in Section 2. In particular, our approach merely simplifies the proof of the asymptotic normality for a fixed corner extracted in the unitary ensemble given in Krishnapur (2009). Further, we show large deviation theorems both for matrix random spectral measure and their infinite dimensional lifting. All proofs are postponed to Section 4.

2. Preliminaries

2.1. Some notations and definitions

To begin with, let us give some definitions and notations. Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathcal{H} = \ell^2_{\mathbb{C}}(\mathbb{N})$. For $i \ge 1$, let $e_i = (0, \ldots, 0, 1, 0, \ldots)$ be the ith element of the canonical basis of $\mathcal{H}$ and for $p \ge 1$ let $\mathcal{H}_p$ be $span\{e_1, \ldots, e_p\}$. We define several sets of matrices with complex entries:

- $\mathbb{M}_{p,n}$, the set of $p \times n$ matrices with complex entries,
- $\mathbb{U}(n)$, the set of $n \times n$ unitary matrices.

At last, I_p denotes the $p \times p$ identity matrix on $\mathcal{H}_p$.