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Journal of Statistical Planning and Inference 136 (2006) 193–198

journal of statistical planning and inference

www.elsevier.com/locate/jspi

On the characterization of distributions by their L-moments

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Received 6 November 2003; accepted 14 June 2004
Available online 25 July 2004

Abstract

A distribution with finite mean is uniquely determined by the set of expectations of the largest (or smallest) order statistics from samples of size $1, 2, \ldots$. However, this characterization contains some redundancy; some of the expectations can be dropped from the set and the remaining elements of the set still suffice to characterize the distribution. The rth L-moment of a distribution is a linear combination of the expectations of the largest (or smallest) order statistics from samples of size $1, 2, \ldots, r$. We show that a wide range of distributions can be characterized by their L-moments with no redundancy; a set that contains all of the L-moments except one no longer suffices to characterize the distribution.

MSC: 62E10; 62G30

Keywords: Legendre polynomials; Order statistics; Quantile function

Let $X_1, \ldots, X_n$ be a random sample from a probability distribution with cumulative distribution function F. When the sample is sorted into ascending order and written as $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$, we call $X_{k:n}$ the kth order statistic.

The expectations of the extreme order statistics characterize a distribution. If $E|X|$ is finite, either of the sets $\{EX_{1:n} : n = 1, 2, \ldots\}$ or $\{EX_{n:n} : n = 1, 2, \ldots\}$ determines F (Chan, 1967; Konheim, 1971).

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0378-3758/$ - see front matter
doi:10.1016/j.jspi.2004.06.004

Though the expectations of extreme order statistics characterize a distribution, they contain redundant information: there are proper subsets of the expectations of extreme order statistics that still suffice to characterize a distribution. For example, Huang (1975) showed that if $EX_{1:n_1}$ is finite and

$$\sum_{i=1}^{\infty} n_i^{-1} = \infty,$$

then the set $\{EX_{1:n} : n = n_1, n_2, n_3, \ldots\}$ determines F. Other conditions under which a set of expectations of order statistics characterizes the distribution are given by Huang (1989) and references therein.

Hosking (1990) defined L-moments to be certain linear combinations of expectations of order statistics. In terms of expectations of extreme order statistics, L-moments can be written as

$$\lambda_r = (-1)^{r-1} \sum_{k=1}^{r} p^*_{r-1,k-1} k^{-1} EX_{1:k} = \sum_{k=1}^{r} p^*_{r-1,k-1} k^{-1} EX_{k:k}.$$

Here,

$$p^*_{r,k} = (-1)^{r-k} \binom{r}{k} \binom{r+k}{k} = \frac{(-1)^{r-k}(r+k)!}{(k!)^2\,(r-k)!}$$

are the coefficients of the shifted Legendre polynomials

$$P^*_r(u) = \sum_{k=0}^{r} p^*_{r,k} u^k.$$

Shifted Legendre polynomials are related to the "ordinary" Legendre polynomials by $P^*_r(u) = P_r(2u-1)$; they are orthogonal polynomials on the interval [0, 1] with constant weight function.

L-moments can be used as summary statistics for data samples, and to identify probability distributions and fit them to data. A brief description is given in Hosking (1998). L-moments are now widely used in hydrology to summarize data and fit flood frequency distributions: recent examples include Kjeldsen et al. (2002), Kroll and Vogel (2002), Lim and Lye (2003) and Zaidman et al. (2003). In other recent work, Karvanen et al. (2002) used L-moments for fitting distributions in independent component analysis in signal processing, and Jones and Balakrishnan (2002) pointed out some relationships between integrals occurring in the definition of moments and L-moments. Some generalizations of L-moments have been defined by Elamir and Seheult (2003).

The L-moments are determined by the expectations of extreme order statistics, and vice versa. The set of L-moments therefore shares with the set of expectations of extreme order statistics the property of determining the distribution. We shall now show that, for a wide range of distributions, the characterization by L-moments is nonredundant, in that if even one L-moment is dropped from the set the remaining L-moments no longer suffice to determine the distribution.