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Short communication

Generalizing Koenker's distribution

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ABSTRACT

Koenker (1993) discovered an interesting distribution whose α quantile and α expectile coincide for every α in $(0,1)$. We analytically characterize the distribution whose $\omega(\alpha)$ expectile and α quantile coincide, where $\omega(\cdot)$ can be any monotone function. We further apply the general theory to derive generalized Koenker's distributions corresponding to some simple mapping functions. Similar to Koenker's distribution, the generalized Koenker's distributions do not have a finite second moment.

1. Introduction

Let q_α denote the α-quantile of a random variable Z such that $\alpha=\mathrm{pr}(Z\le q_\alpha)$. Koenker and Bassett (1978) showed that q_α is the minimizer of the α-asymmetric least absolute deviation risk

$$q_\alpha=\arg\min_b E\{\psi_\alpha(Z-b)\}, \tag{1}$$

where $\psi_\alpha(t)=(1-\alpha)|t|I(t\le 0)+\alpha|t|I(t>0)$. Koenker and Bassett (1978) showed that one can use the above result to derive regression quantiles. For example, assume that the conditional α quantile of a response variable Y given covariates $X=x$ is a linear function of x, i.e., $\alpha=\mathrm{pr}(Y\le x^{\mathrm{T}}\beta+c|X=x)$. Given a random sample (y_i,x_i), $1\le i\le n$, quantile regression (Koenker and Bassett, 1978) estimator of (β,c) is defined as

$$\arg\min_{\beta,c} n^{-1}\sum_{i=1}^{n}\psi_\alpha(y_i-x_i^{\mathrm{T}}\beta-c).$$

The ω expectile of a random variable Z, denoted by f_ω, is defined as (Newey and Powell, 1987)

$$\omega=\frac{E\{|Z-f_\omega|I(Z\le f_\omega)\}}{E\{|Z-f_\omega|\}}. \tag{2}$$

Newey and Powell (1987) showed that f_ω is the minimizer of the asymmetric least squares risk (provided that the second moment is finite), i.e.,

$$f_\omega=\arg\min_b E\{\phi_\omega(Z-b)\}, \tag{3}$$

where $\phi_\omega(t)=(1-\omega)t^2I(t\le 0)+\omega t^2I(t>0)$. Newey and Powell (1987) further developed expectile regression based on (3). Assume that the conditional ω expectile of Y given $X=x$ is a linear function of x, then given a random sample

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$(y_i, x_i),\ 1 \le i \le n$, expectile regression estimates the linear expectile function by

$$\arg\min_{\beta,c} n^{-1} \sum_{i=1}^{n} \phi_\alpha(y_i - x_i^{\mathrm{T}}\beta - c).$$

See also Efron (1991).

Quantile regression includes the median regression as a special case ($\alpha = 0.5$), while expectile regression includes the mean regression as a special case ($\omega = 0.5$). Both methods use asymmetric weights to penalize negative and positive residuals. By doing so, one can explore the conditional distribution of the response variable given covariates. This is the main advantage of expectile/quantile regression over the conventional mean/median regression. Quantile/expectile regression have found many applications in statistics and econometrics (Koenker, 2005; Yao and Tong, 1996; De Rossi and Harvey, 2009; Taylor, 2008). Breckling and Chambers (1988) developed a more general M-quantile theory.

Koenker (1992) proposed the following interesting question: for which distribution its α quantile and α expectile coincide for every $\alpha \in (0,1)$? The answer was given in Koenker (1993). The α quantile must be

$$q_\alpha = \frac{2\alpha - 1}{\alpha^{1/2}(1-\alpha)^{1/2}} \tag{4}$$

or any affine transformation thereof. Here we further generalize Koenker's result in the following way. Consider a deterministic function of α mapping from (0,1) to (0,1), denoted by $\omega(\alpha)$. We find the solution to the following question:

Given a (proper) function $\omega(\alpha)$, which distributions have matching $\omega(\alpha)$-expectile and α-quantile for every $\alpha \in (0,1)$?

Koenker's original problem corresponds to the case when $\omega(\alpha) = \alpha$. We provide the answer in Section 2. Moreover, to follow Koenker's idea of examining the relation between expectile and quantile, we are particularly interested in finding out for what distributions the mapping function is simple. In Section 3 we work out some interesting distributions corresponding to simple mapping functions: (1) $(1-\omega)/\omega = c(1-\alpha)/\alpha,\ c > 0$, (2) $\omega(\alpha) = \alpha^2$, and (3) $\omega(\alpha) = \alpha^{1/2}$. These distributions are named generalized Koenker's distributions.

2. Theory

Theorem 1. *Let Z be a random variable following a distribution such that its $\omega(\alpha)$ expectile and α quantile coincide for every $\alpha \in (0,1)$, where $\omega(\cdot)$ is a monotone increasing function and $\omega(0) = 0,\ \omega(1) = 1$. There exists some α_0 such that $\omega(\alpha_0) = 1/2$. Define*

$$S(\alpha) = \frac{2\omega(\alpha) - 1}{2\omega(\alpha)\alpha - \alpha - \omega(\alpha)}. \tag{5}$$

Then the α quantile of Z is given by

$$q_\alpha = -S(\alpha) \exp\left\{ \int_{\alpha_0}^{\alpha} S(t)\, dt \right\}, \tag{6}$$

or any affine transformation thereof.

Theorem 1 characterizes the distribution via an explicit expression for its quantile functions. The cumulative distribution function of Z, denoted by $F(z)$, is the inverse function of q_α and the density function is $f(z) = (\dot{q}_{F(z)})^{-1}$, provided $\dot{q_\alpha}$ exists. Here $\dot{q_\alpha}$ denotes the derivative of q_α with respective to α. Eq. (6) is sufficient for simulating the distribution: first generate U from the uniform distribution on (0,1), then q_U follows the distribution in Theorem 1.

Before giving the proof of Theorem 1, let us first revisit Koenker's problem to demonstrate Theorem 1. When $\omega(\alpha) = \alpha$ we have $S(\alpha) = (2a-1)/(2a^2 - 2a)$. Obviously, $\alpha_0 = 1/2$ and $\int_{0.5}^{\alpha} S(t)\, dt = 0.5 \log(4\alpha(1-\alpha))$. Then by (6) we have

$$q_\alpha = -S(\alpha) \exp\left\{ \int_{0.5}^{\alpha} S(t)\, dt \right\} = \frac{2\alpha - 1}{\alpha^{1/2}(1-\alpha)^{1/2}}$$

or any affine transformation thereof.

Proof. Let Z be the random variable and its $\omega(\alpha)$ expectile and α quantile coincide for every $\alpha \in (0,1)$. Obviously, the quantile–expectile matching property is shared by any affine transformation of Z. Without loss of generality we can assume $E(Z) = 0$. Then $q_{\alpha_0} = EZ = 0$. Denote its cumulative distribution function by $G(z)$. First, by letting $f_\omega = q_\alpha$ in (2) we obtain

$$\omega(\alpha) = \frac{E\{|Z - q_\alpha| I_{\{Z \le q_\alpha\}}\}}{E\{|Z - q_\alpha|\}}.$$

By straightforward calculation we have

$$\omega(\alpha) = \frac{\alpha q_\alpha - \int_{-\infty}^{q_\alpha} z\, dG(z)}{2[\alpha q_\alpha - \int_{-\infty}^{q_\alpha} z\, dG(z)] - q_\alpha}.$$