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Parameter estimation for fractional Poisson processes

Dexter O. Cahoy [a], Vladimir V. Uchaikin [b], Wojbor A. Woyczynski [c,*]

[a] Department of Mathematics and Statistics, Louisiana Tech University, USA
[b] Department of Theoretical and Mathematical Physics, Ul'yanovsk State University, Russia
[c] Department of Statistics, and Center for Stochastic and Chaotic Processes in Science and Technology, Case Western Reserve University, USA

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ABSTRACT

The paper proposes a formal estimation procedure for parameters of the *fractional Poisson process* (fPp). Such procedures are needed to make the fPp model usable in applied situations. The basic idea of fPp, motivated by experimental data with long memory is to make the standard Poisson model more flexible by permitting non-exponential, heavy-tailed distributions of interarrival times and different scaling properties. We establish the asymptotic normality of our estimators for the two parameters appearing in our fPp model. This fact permits construction of the corresponding confidence intervals. The properties of the estimators are then tested using simulated data.

1. Introduction

The paper proposes a formal estimation procedure for parameters of the *fractional Poisson process* (fPp). Such procedures are needed to make the fPp model usable in applied situations. Different versions of fPp have been studied recently by several authors, see, in particular, Repin and Saichev (2000), Jumarie (2001), Laskin (2003), Wang and Wen (2003), and Wang et al. (2006), so we start our exposition from the basic definitions to make it clear which stochastic model we are working with. Some of the preliminary results on this model appeared in Cahoy (2007), and Uchaikin et al. (2008) but we restate them in the first couple of sections for the sake of completeness of presentation. The basic idea of fPp, motivated by experimental data with long memory (such as some network traffic, neuronal firings, and other signals generated by complex systems), is to make the standard Poisson model more flexible by permitting non-exponential, heavy-tailed distributions of interarrival times. However, the price one has to pay for such flexibility is loss of the Markov property, a similar situation to that encountered in the case of certain anomalous diffusions, see, e.g., Piryatinska et al. (2005). To partly replace this loss one demands some scaling properties of the interarrival times' distributions which makes other tools available; in this paper they are the fractional calculus and the link between fractional Poisson process and $\alpha-$stable Lévy densities. Based on the latter connection we establish the asymptotic normality of our estimators for the two parameters appearing in our fPp model: the intensity rate μ, and the fractional exponent ν. This fact permits construction of the corresponding confidence intervals. The properties of the estimators are then tested using synthetic data.

The paper is composed as follows: Section 2 introduces the basic definition of fPp and the fractional calculus tools needed to study it. Section 3 proves the basic structural theorem relating fPp to $\alpha-$stable Lévy random variables which

* Corresponding author. Tel.: +1 216 368 6942.
E-mail address: waw@case.edu (W.A. Woyczynski).

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makes efficient simulation of the former possible. In Section 4, we describe nontrivial scaling limits of the marginal distributions of fPp. Section 5 introduces the concept of the method-of-moments estimators in the fPp context and calculates them. They are proven asymptotically normal in Section 6. Finally, we test our procedures numerically on simulated data in Section 7. The concluding remarks in Section 8 are then followed by two brief appendices, one on α^+-stable densities, and one on an alternative fPp model.

2. FPp interarrival time

The fractional Poisson process $N_\nu(t)$, $0<\nu\leq 1$, $t>0$, was defined in Repin and Saichev (2000) via the following formula for the Laplace transform of the p.d.f. $\psi_\nu(t)$ of its i.i.d. interarrival times T_i, $i=1,2,\ldots$:

$$\{\mathbf{L}\psi_\nu(t)\}(\lambda)\equiv\tilde{\psi}_\nu(\lambda)\equiv\int_0^\infty e^{-\lambda t}\psi_\nu(t)\,dt=\frac{\mu}{\mu+\lambda^\nu}, \tag{1}$$

where $\mu>0$ is a parameter. For $\nu=1$, the above transform coincides with the Laplace transform

$$\tilde{\psi}_1(\lambda)=\frac{\mu}{\mu+\lambda}$$

of the exponential interarrival time density of the ordinary Poisson process with parameter $\mu=\mathbf{E}N_1(1)$.

Using the inverse Laplace transform the above cited authors derived the *singular integral equation* for $\psi_\nu(t)$:

$$\psi_\nu(t)+\frac{\mu}{\Gamma(\nu)}\int_0^t\psi_\nu(\tau)\frac{d\tau}{[\mu(t-\tau)]^{1-\nu}}=\frac{\mu^\nu}{\Gamma(\nu)}t^{\nu-1},$$

which is equivalent to the *fractional differential equation*

$${}_0D_t^\nu\psi_\nu(t)+\mu\psi_\nu(t)=\delta(t),$$

where the Liouville derivative operator ${}_0D_t^\nu=d^\nu/dt^\nu$ (see, e.g., Kilbas et al., 2006) is defined via the formula

$${}_0D_t^\nu\psi_\nu(\tau)=\frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t\psi_\nu(\tau)\frac{d\tau}{[\mu(t-\tau)]^{1-\nu}}.$$

These characterizations permitted them to obtain the following integral representation for the p.d.f. $\psi_\nu(t)$:

$$\psi_\nu(t)=\frac{1}{t}\int_0^\infty e^{-x}\phi_\nu(\mu t/x)\,dx, \tag{2}$$

where

$$\phi_\nu(\xi)=\frac{\sin(\nu\pi)}{\pi[\xi^\nu+\xi^{-\nu}+2\cos(\nu\pi)]},$$

and demonstrate that the tail probability distribution of the waiting time T is of the form

$$\mathbf{P}(T>t)=\int_t^\infty\psi_\nu(\tau)\,d\tau=E_\nu(-\mu t^\nu), \tag{3}$$

where

$$E_\nu(z)=\sum_{n=0}^\infty\frac{z^n}{\Gamma(\nu n+1)} \tag{4}$$

is the Mittag–Leffler function (see, e.g., Kilbas et al., 2006).

Remark 2.1. Observe that the Mittag–Leffler function is a fractional generalization of the standard exponential function $\exp(z)$; indeed $E_1(z)=\exp(z)$. It has been widely used to describe probability distributions appearing in finance and economics, anomalous diffusion, transport of charge carriers in semiconductors, and light propagation through random media (see, e.g., Piryatinska et al., 2005; Uchaikin and Zolotarev, 1999).

In view of (3)–(4), the interarrival time density for the fractional Poisson process can be easily shown to be

$$\psi_\nu(t)=\mu t^{\nu-1}E_{\nu,\nu}(-\mu t^\nu),\qquad t\geq 0, \tag{5}$$

where

$$E_{\alpha,\beta}(z)=\sum_{n=0}^\infty\frac{z^n}{\Gamma(\alpha n+\beta)}$$

is the generalized, two-parameter Mittag–Leffler function. Also, the above information automatically gives the p.d.f.

$$f_n^\nu(t)=\mu^n\nu\frac{t^{\nu n-1}}{(n-1)!}E_\nu^{(n)}(-\mu t^\nu) \tag{6}$$