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Two-level minimum aberration designs in $N = 2 \pmod 4$ runs

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ABSTRACT

For two-level factorials, we consider designs in $N=2 \pmod 4$ runs as obtained by adding two runs, with a certain coincidence pattern, to an orthogonal array of strength two. These designs are known to be optimal main effect plans in a very broad sense in the absence of interactions. Among them, we explore the ones having minimum aberration, with a view to ensuring maximum model robustness even when interactions are possibly present. This is done by sequentially minimizing a measure of the bias caused by interactions of successively higher orders.

1. Introduction

Minimum aberration (MA) designs have been of significant interest from the perspective of model robustness in factorial experiments; see Mukerjee and Wu (2006) and Xu et al. (2009) for reviews. We focus on two-level factorials where two-symbol orthogonal arrays (OAs) play a key role in the study of MA designs. An OA$(n, m, 2, 2)$ of strength two, where $n=4t$, is an $n \times m$ array with entries say ± 1, such that all four possible pairs of symbols occur equally often as rows in every $n \times 2$ subarray. The rows of the OA, interpreted as experimental runs, represent a fraction of a 2^m factorial.

A rich literature on MA designs is now available for two-level factorials when the run size n is a multiple of four. As noted in Tang and Deng (1999), such an MA design sequentially minimizes, among designs given by an OA$(n, m, 2, 2)$, a certain measure of the bias, due to interactions of successively higher orders, in the estimation of main effects. This conforms to the effect hierarchy principle (Wu and Hamada (2009), p. 172) and makes perfect statistical sense because (a) designs based on OA$(n, m, 2, 2)$ are universally optimal main effect plans when interactions are absent (Cheng (1980a)), while (b) sequential minimization of bias among these designs maximizes model robustness even in the presence of interactions. Recently, Zhang and Mukerjee (2013, hereafter abbreviated ZM) extended this approach to find MA designs with run size $1 \pmod 4$ via sequential minimization of the bias due to interactions, among designs obtained by adding any single run to an OA$(n, m, 2, 2)$ – in the spirit of (a), all competing designs are again optimal for the general mean and main effects, in a very broad sense (Cheng (1980b)), when interactions are absent.

The present paper aims at exploring MA designs when the run size equals $2 \pmod 4$. Let $n=4t$, $t \geq 1$, and $m \geq 3$. With the same motivation as in (a) and (b) above, we consider the class C of $N(=n+2)$-run designs obtained by adding two runs to an OA$(n, m, 2, 2)$ such that the two added runs have $\lceil m/2 \rceil$ coincidences, i.e., have the same entry in $\lceil m/2 \rceil$ positions, and then find an MA design in C by sequentially minimizing the bias, due to interactions of successively higher orders, in the estimation of main effects. Here $\lceil m/2 \rceil$ is the largest integer in $m/2$. Consideration of the class C is justified because,

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following Cheng (1980b) and Jacroux et al. (1983), in the absence of interactions any design in C is optimal, among all N-run designs, for estimating the general mean and main effects in a very broad sense that includes the popular D-, A- and E-criteria. An MA design in C, on the other hand, ensures maximum protection against bias when the true model possibly includes interactions.

The obvious nonorthogonality of the designs in C even in the absence of interactions complicates our task. Compared to ZM, the challenge is more serious because now two added runs, rather than one, have to be handled jointly, subject to a constraint on their number of coincidences. In particular, we will need to control simultaneously the scalar products of the two added runs with those in the OA for obtaining theoretical results. After introducing the MA criterion in the present context in Section 2, we report MA designs in Section 3, followed by some concluding remarks in Section 4. All proofs appear in the appendix. In order to save space, details are omitted in some places where the findings in ZM facilitate the deduction.

2. Minimum aberration criterion

Consider a 2^m factorial in factors $F_1, \ldots, F_m$, each at levels ± 1. Let $\tau(j_1 \ldots j_m)$ be the treatment effect of a typical treatment combination $j_1 \ldots j_m$, where $j_i = -1$ or 1, $1 \le i \le m$. Under the usual orthogonal parametrization and a full factorial model,

$$\tau(j_1 \ldots j_m) = \sum_{x \in \Omega} j_1^{x_1} \ldots j_m^{x_m} \beta(x), \tag{1}$$

for each $j_1 \ldots j_m$, where Ω is the set of binary m-tuples. For $x = x_1 \ldots x_m \in \Omega$, the parameter $\beta(x)$ in (1) represents the general mean if $x = 0 \ldots 0$, or the factorial effect $F_1^{x_1} \ldots F_m^{x_m}$ otherwise. In particular, if all interactions are absent, then (1) reduces to

$$\tau(j_1 \ldots j_m) = \beta_0 + j_1\beta_1 + \cdots + j_m\beta_m, \tag{2}$$

where we write $\beta_0 = \beta(0 \ldots 0)$, $\beta_1 = \beta(10 \ldots 0)$, etc. for notational simplicity.

Let $Q = (q_{ui})$, $1 \le u \le n$, $1 \le i \le m$, be an OA$(n, m, 2, 2)$, and consider a design in C obtained by adding two runs $q_{u1} \ldots q_{um}$ $(u = n+1,\ n+2)$, with $\lceil m/2 \rceil$ coincidences, to Q. Denote the observational vector by $Y = (Y_1, \ldots, Y_N)^T$, $N = n+2$, where $Y_1, \ldots, Y_n$ arise from the runs given by the rows of Q, and Y_{n+1} and Y_{n+2} from the two added runs. Here T stands for transposition. Write $q_u = (q_{u1}, \ldots, q_{um})^T$, $1 \le u \le N$. Each q_u has elements ± 1 and $q_1^T, \ldots, q_n^T$ are the n rows of Q. Then by (2), in the absence of interactions,

$$E(Y) = Z\beta^*, \tag{3}$$

where

$$Z = \begin{bmatrix} 1_n^T & 1 & 1 \\ Q^T & q_{n+1} & q_{n+2} \end{bmatrix}^T, \quad \beta^* = \begin{pmatrix} \beta_0 \\ \beta \end{pmatrix}, \tag{4}$$

$\beta = (\beta_1, \ldots, \beta_m)^T$ is the vector of the main effect parameters and 1_n is the $n \times 1$ column vector of ones. The observations Y_u, $1 \le u \le N$, are supposed to be homoscedastic and uncorrelated.

Under the model (3) arising when interactions are absent, the best linear unbiased estimator of β equals $\hat{\beta} = LY$, where L is the $m \times N$ submatrix of $(Z^TZ)^{-1}Z^T$ given by the last m rows of the latter. In order to show L explicitly, we write I_m for the identity matrix of order m, and note that

$$Q^T 1_n = 0, \quad Q^TQ = nI_m, \quad N = n+2, \quad q_u^Tq_u = m \ \ (1 \le u \le N). \tag{5}$$

The first two relationships in (5) hold because Q is an OA$(n, m, 2, 2)$ with elements ± 1. Also, let

$$a = q_{n+1}^T q_{n+2}, \quad b_1 = n + m - a, \quad b_2 = N + m + a, \quad b_0 = nb_1b_2, \tag{6}$$

$$\xi_1 = \tfrac{1}{2}(q_{n+1} + q_{n+2}), \quad \xi_2 = \tfrac{1}{2}(q_{n+1} - q_{n+2}), \quad \pi_1 = Q\xi_1, \quad \pi_2 = Q\xi_2. \tag{7}$$

By (5)–(7),

$$\xi_1^T\xi_1 = \tfrac{1}{2}(m+a), \quad \xi_2^T\xi_2 = \tfrac{1}{2}(m-a), \quad \xi_1^T\xi_2 = 0 \tag{8}$$

Now, from (4) and (5)–(8), after a long algebra that involves explicit evaluation of $(Z^TZ)^{-1}$,

$$L = b_0^{-1}[L_0 \quad l_1 \quad l_2], \tag{9}$$

where the $m \times n$ matrix L_0 and the $m \times 1$ vectors l_1, l_2 are given by

$$L_0 = (b_1b_2I_m - 2b_1\xi_1\xi_1^T - 2b_2\xi_2\xi_2^T)Q^T - 2b_1\xi_1 1_n^T, \quad l_1 = n(b_1\xi_1 + b_2\xi_2), \quad l_2 = n(b_1\xi_1 - b_2\xi_2). \tag{10}$$

For subsequent use, we also note that by (6)–(8) and (10), on simplification,

$$L_0^TL_0 = 2(m+a)b_1^2 1_n 1_n^T - 2Nb_1^2(1_n\pi_1^T + \pi_1 1_n^T) + b_1^2b_2^2QQ^T - 2b_1^2(b_2+N)\pi_1\pi_1^T - 2b_2^2(b_1+n)\pi_2\pi_2^T,$$

$$L_0^Tl_1 = n\{Nb_1^2\pi_1 + nb_2^2\pi_2 - (m+a)b_1^2 1_n\},$$
$$L_0^Tl_2 = n\{Nb_1^2\pi_1 - nb_2^2\pi_2 - (m+a)b_1^2 1_n\},$$