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# Characteristics of some classes of space–time covariance functions

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ABSTRACT

Although a wide list of classes of space–time covariance functions is now available, selecting an appropriate class of models for a variable under study is still difficult and it represents a priority problem with respect to the choice of a particular model of a specified class. Then, knowing the characteristics of various classes of covariances, and their auxiliary functions, and matching those with the characteristics of the empirical space–time covariance surface might be helpful in the selection of a suitable class. In this paper some characteristics, such as behavior at the origin, asymptotic behavior, nonseparability and anisotropy aspects, are studied for some well known classes of covariance models of stationary space–time random fields. Moreover, some important issues related to modeling choices are described and a case study is presented.



## 1. Introduction

Nowadays, some systematic contributions on space-time geostatistical analysis can be found in the literature (Cressie and Wikle, 2011; Christakos, 2012) and various classes of space–time covariance functions are available. The metric class of models (or geometric anisotropy models), where the anisotropy factors between the space and time axes are introduced, has represented one of the first attempts to construct parametric families of space–time covariance functions (Dimitrakopoulos and Luo, 1994). Essentially, some other space–time covariance models have been built by assuming separability between space and time, such as the sum model (Rouhani and Hall, 1989) and the product model. The latter, whose strict positive definiteness is explored in De Iaco et al. (2011b), is known in two versions: the one obtained by the product of purely spatial and purely temporal covariance functions (Posa, 1993) and the other one where the global variance of the random field is multiplied by a purely spatial correlation function and a purely temporal correlation function (Haas, 1995). Then, the above classes have been often used as a starting point for deriving families of nonseparable space–time covariance models through appropriate mixture procedures, as also shown in Stein (1986), De Iaco et al. (2001, 2002) and Ma (2002, 2003). In addition, various classes of nonseparable space–time covariances have been constructed through different approaches by Cressie and Huang (1999), Gneiting (2002), Kolovos et al. (2004), Stein (2005), Ma (2005), Porcu and Mateu (2007), Porcu et al. (2008), Rodrigues and Diggle (2010), among others. Moreover, other contributions deal with nonseparable anisotropic space–time covariance models (Fernandez-Casal et al., 2003; Stein, 2005; Porcu et al., 2006) and nonseparable asymmetric space–time covariance models (Porcu et al., 2006; Gneiting et al., 2007; Jun and Stein, 2007). The selection of an appropriate class of models might be based on its geometric features and theoretical properties. Indeed, one might

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look for the class of models whose properties are consistent with respect to the characteristics of the empirical space–time covariance surface estimated from the data. For this reason, analyzing the following questions is essential: (a) how do the spatial and/or the temporal marginals behave at the origin? (b) does the space–time data set present different variability along space and time? (c) which type of nonseparability is highlighted by the data? (d) which kind of spatial anisotropy is required by the data? Note that the following analytical study regards some well known classes of models, not just some specific models. Hence, the analyzed features allow different classes to be characterized and properly selected according to the data, even before defining the specific model suitable to describe the correlation structure of the variable under study.

In this paper, the theoretical framework of some well-known classes of fully symmetric covariance functions for stationary space–time random fields (*STRF*) is given in Section 2 and some characteristics of the selected classes as well as some examples are discussed in Section 3. Finally, in Section 4, some aspects related to the modeling choices are presented. In Section 5, the use of some results is illustrated through a case study regarding an environmental data set.

## 2. Classes of space–time covariances

Let $\{Z(\mathbf{s},t),(\mathbf{s},t)\in D\times T\}$ be a second order stationary *STRF*, where $D\subseteq\mathbb{R}^d$ is the spatial domain and $T\subseteq\mathbb{R}$ the temporal one, with zero expected value and covariance $C_{ST}(\mathbf{h}_s,h_t)=E[Z(\mathbf{s},t)Z(\mathbf{s}',t')]$, with $\mathbf{h}_s=(\mathbf{s}-\mathbf{s}')$ and $h_t=(t-t')$. In modeling space–time covariance functions, one should ensure that the model is positive definite. The property of positive definiteness is a necessary and sufficient condition for a function to be a covariance (Yaglom, 1987; De Iaco et al., 2011b), which, under second order stationarity assumption, implies that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j C_{ST}(\mathbf{s}_i-\mathbf{s}_j,t_i-t_j)\geq 0, \tag{1}$$

for any $n\in N$ and any choice of $(\mathbf{s}_1,t_1),\ldots,(\mathbf{s}_n,t_n)\in\mathbb{R}^d\times\mathbb{R}$ and $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$. If the quadratic form (1) is strictly positive for any choice of distinct points $(\mathbf{s}_1,t_1),\ldots,(\mathbf{s}_n,t_n)\in\mathbb{R}^d\times\mathbb{R}$ and $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$ not all zero, then $C_{ST}$ is *strictly positive definite*. Strict positive definiteness is essential for spatial and spatio-temporal prediction, since it ensures the existence of a unique solution of the kriging system.

Various assumptions are often made for covariance functions of stationary *STRF*, such as full symmetry and separability, which can be tested by using different methodologies (Mitchell et al., 2005; Fuentes, 2006; Li et al., 2007). Although a wide dissertation on the smoothness of some covariance functions has been already furnished by Stein (2005) and some characteristics for the Gneiting class have been proposed by Zastavnyi and Porcu (2011), this paper focuses on other interesting features of the following classes of space–time stationary covariance models:

- Cressie–Huang class of models (Cressie and Huang, 1999)
$$C_{ST}(\mathbf{h}_s,h_t)=\int_{\mathbb{R}^d}e^{i\mathbf{h}_s^T\boldsymbol{\omega}}\rho(h_t;\boldsymbol{\omega})k(\boldsymbol{\omega})\,d\boldsymbol{\omega}, \tag{2}$$
where $\rho(\cdot;\boldsymbol{\omega})$ is a continuous integrable correlation function for all $\boldsymbol{\omega}\in\mathbb{R}^d$, and $k(\cdot)$ is a positive function, which is integrable on $\mathbb{R}^d$;
- Gneiting class of models (Gneiting, 2002)
$$C_{ST}(\mathbf{h}_s,h_t)=\frac{\sigma^2}{[\psi(h_t^2)]^{d/2}}\phi\left(\frac{\|\mathbf{h}_s\|^2}{\psi(h_t^2)}\right), \tag{3}$$
where $\phi(t),t\geq 0$, is a completely monotone function and $\psi(t),t\geq 0$ is a positive function with completely monotone derivative;
- the class of integrated models (De Iaco et al., 2002)
$$C_{ST}(\mathbf{h}_s,h_t)=\int_V[k_1C_S(\mathbf{h}_s;x)C_T(h_t;x)+k_2C_S(\mathbf{h}_s;x)+k_3C_T(h_t;x)]\,d\mu(x), \tag{4}$$
where $\mu(x)$ is a positive measure on $U\subseteq\mathbb{R}$, $C_S(\mathbf{h}_s;x)$ and $C_T(h_t;x)$ are covariance functions defined on $D\subseteq\mathbb{R}^d$ and $T\subseteq\mathbb{R}$, respectively, for all $x\in V\subseteq U$, $k_1>0$, $k_2,k_3\geq 0$. If $k_2=k_3=0$, the above class of models is called integrated product models, otherwise it is called integrated product–sum models.

Even though the above selection is not supposed to be exhaustive, the proposed classes can describe most of the applications, because of their complementary properties as well as their similarities and differences, specified hereafter.

- These models are nonseparable, but they have been built using different methods and covariance properties; hence, they have their own characteristics, in terms of the possibility of identifying the space–time interaction parameters and the connection with the corresponding marginals;
- each of the above mentioned classes is not a sub-class of the others. For example, the integrated product models, which are generally non-integrable, cannot be obtained in general from the Cressie–Huang representation (Cressie and Huang,