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Almost sure behaviour of near moving maxima

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ABSTRACT

Let $\{X_n\}$ be a sequence of independent and identically distributed random variables defined over a common probability space $(\Omega,\mathcal{F},\mathcal{P})$ with common continuous distribution function F. Define $\eta_n = \max_{n-a_n<j\le n} X_j$, where a_n is an integer with $0<a_n<n, n>1$. For any constant $a>0$, let $K_n^{(m)}(a) = \#\{j, n-a_n<j\le n, X_j \in (\eta_n - a, \eta_n]\}, n>1$. Then $K_n^{(m)}(a)$ denotes the number of observations near moving maxima. In this paper, we obtain conditions for $(K_n^{(m)}(a))$ to converge to 1 almost surely (a.s.), when $a_n=[n^p]$ and $a_n=[pn], 0<p<1, n\ge 1$.

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1. Introduction

Let $\{X_n\}$ be a sequence of independent and identically distributed (i.i.d.) random variables (r.v.), defined over a common probability space $(\Omega,\mathcal{F},\mathcal{P})$, with common continuous distribution function (d.f.) F. Let for $n\ge 2, \{a_n,\ 0<a_n<n\}$ be a sequence of integers. Define $M_n=\max_{1\le j\le n} X_j$, and $\eta_n = \max_{n-a_n<j\le n} X_j$, $n\ge 2$. It is well known that M_n is the partial maxima. In the same spirit η_n is called the moving maxima. Rothmann and Russo (1991) introduced the concept of moving maxima for studying the asymptotic behaviour of the extremes, either when some of the initial observations are not available or when some of the initial observations are not properly recorded. In this paper, the near moving maxima is introduced and some asymptotic properties are studied. For any $a>0$, define $K_n(a) = \#\{j, 1\le j\le n,\ X_j \in (M_n - a, M_n]\}$ and $K_n^{(m)}(a) = \#\{j, n-a_n<j\le n,\ X_j\in(\eta_n - a,\ \eta_n]\}$ Then $(K_n(a))$ is known as the sequence of near-maxima. We call $(K_n^{(m)}(a))$ as the sequence of near-moving maxima since, $(K_n^{(m)}(a))$ gives the number of observations among $X_{n-a_n+1}, X_{n-a_n+2},\dots,X_n$ in the neighbourhood $(\eta_n - a,\ \eta_n]$. One may observe that $K_n^{(m)}(a) \stackrel{d}{=} K_{a_n}(a), n\ge 2$, where $X\stackrel{d}{=}Y$ means X and Y are distributionally equal. The asymptotic study of $(K_n(a))$ has drawn the attention of workers over the past two decades. In the study of insurance claim models $(K_n(a))$ is the number of claims in a neighbourhood of M_n; in the study of heights of waves, it gives the number of waves above a random threshold level $(M_n - a)$. In queueing theory, it gives the number of customers, whose service times are close to that of the most demanding customer. Pakes and Steutel (1997) were the first to initiate the study of the asymptotic properties of $(K_n(a))$. Let $r_F = \sup\{x; F(x)<1\}$. When $r_F=\infty$, they introduced the following tail thickness condition in terms of $\overline{F} = 1-F$. For any $a>0$, let there exist a constant $\gamma(a), 0\le\gamma(a)\le 1$, such that

$$0 \le \lim_{x\to\infty} \frac{\overline{F}(x) - \overline{F}(x+a)}{\overline{F}(x)} = \gamma(a) \le 1. \tag{1.1}$$

They classified the d.f.s satisfying (1.1) as thick tailed when $\gamma(a)=0$, medium tailed when $0<\gamma(a)<1$ and thin tailed when $\gamma(a)=1$ and established that $K_n(a)\xrightarrow{p} 1$ ($\xrightarrow{p}$ means convergence in probability), when $\gamma(a)=0; (K_n(a))$ converges to a

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geometric distribution with values $1,2,3,\ldots$ (shifted), when $0<\gamma(a)<1$ and $K_n(a)\xrightarrow{p}\infty$ whenever $\gamma(a)=1$. One may observe that the class of all d.f.s with regularly varying right tail (such as Pareto, Stable etc) are thick tailed, exponential distribution is medium tailed and the normal distribution is thin tailed. It is interesting to see that among the d.f.s with Weibullian right tail, described by $1-F(x)=e^{-cx^{\alpha}}x^{\beta}(1+o(1))$ as $x\to\infty$, $c>0$, $\alpha>0$, $-\infty<\beta<\infty$, F will be thick tailed when $0<\alpha<1$, medium tailed when $\alpha=1$ and thin tailed when $\alpha>1$. Pakes (2000) observes that whenever $1-F(.)$ is Weibullian and thick tailed ($0<\alpha<1$), (M_n), properly normalized, converges weakly to a Gumbel law. Also, when $\alpha=1$, $\beta=0$ (double exponential) and $\alpha=2$, $\beta=-1$ (normal), it is well known that (M_n), properly normalized, converges to Gumbel law. As such, it is interesting to see that asymptotically $(K_n^{(m)}(a))$ behaves differently, even though (M_n), properly normalized, converges to the same limit law.

Since $K_n^{(m)}(a)\stackrel{d}{=}K_{a_n}(a)$, one may note that all the results of Pakes and Steutel (1997) hold good for $(K_n^{(m)}(a))$ as $a_n\to\infty$. When $K_n^{(m)}(a)\xrightarrow{p}1$ (F is thick tailed), a question that one would naturally ask is, whether $K_n^{(m)}(a)\to 1$ a.s.. In the case of $K_n(a)$, Li (1999) has obtained a criteria for $K_n(a)\to 1$ a.s., as $n\to\infty$, and Pakes (2004) has given criteria for complete convergence and α-stability. In this paper, we obtain criteria for $K_n^{(m)}(a)\to 1$ a.s. as $n\to\infty$, when (i) $a_n=[n^p]$, $0<p<1$, and (ii) $a_n=[pn], 0<p<1$, where for any $\lambda>0$, $[\lambda]$ means the greatest integer less than or equal to λ. In the case of $a_n=[pn], 0<p<1$, our criteria coincides with that of Li (1999). However, when $a_n=[n^p]$, $0<p<1$, the condition is more straight. This may be expected, in view of the fact that the a.s. behaviour of (η_n) differs from that of (M_n), see for example, Rothmann and Russo (1991), Vasudeva (1999), Vasudeva and Srilakshminarayana (2009). We also give criteria for the existence of moments of the number of boundary crossings, whenever $K_n^{(m)}(a)\to 1$ a.s.. Further, we relate the moment criteria to complete convergence and α-stability.

Our findings throw light on many Statistical models. For instance, in non-life insurance models with thick tailed distributions, one may be interested in knowing the number of claims in a neighbourhood of the maximal claim. Similarly, in the study of the transmission time of World Wide Web from the server to client, Crovella et al. (1998) have shown that the transmission time follows a Pareto distribution, which is thick tailed. Here again, a problem of interest is to know the number of transmission times that are close to the maximal transmission time. In such situations, if some of the initial observations are either not recorded properly or missing, the criteria obtained below will help in knowing whether almost surely, no other value (claim size/ transmission time) is close to maximum, which will be a fairly useful information. We present below the criteria by Li (1999), for $(K_n(a))$ to converge to 1 a.s., in view of its importance, in the context of this paper.

Theorem A. *$K_n(a)\to 1$ a.s. if and only if $\int_{-\infty}^{\infty}((F(x+a)-F(x-a))/(1-F(x-a))^2)\,dF(x)<\infty$.*

In the next section, we obtain criteria for $(K_n^{(m)}(a))$ to converge to 1 a.s. In Section 3, the finiteness of the moments of total number of boundary crossings are discussed and in Section 4, the complete convergence and α-stability of $(K_n^{m}(a))$ are studied.

2. Almost sure convergence of $K_n^{(m)}(a)$

In this section, assuming that F is thick tailed, we establish the following a.s. convergence results. Since $K_n^{(m)}(a)\stackrel{d}{=}K_{a_n}(a)$, we note that when F is thick tailed, $K_n^{(m)}(a)\xrightarrow{p}1$, as $a_n\to\infty$.

Theorem 2.1. *Let $a_n=[n^p], 0<p<1, n\geq 1$. Then $K_n^{(m)}(a)\to 1$ a.s. whenever $\int_{-\infty}^{\infty}((F(x+a)-F(x-a))/(1-F(x-a))^{1+1/p})\,dF(x)<\infty$.*

Theorem 2.2. *Let $a_n=[pn], 0<p<1, n\geq 1$. Then $K_n^{(m)}(a)\to 1$ a.s. whenever $\int_{-\infty}^{\infty}((F(x+a)-F(x-a))/(1-F(x-a))^2)\,dF(x)<\infty$.*

Remark 2.1. From the above criteria, observe that for $a_n=[pn], 0<p<1$, the criteria do not depend on p, and in fact coincides with that of Li (1999). When $a_n=[n^p], 0<p<1, p$ has a role and the criteria become stringent.

Theorem 2.3. *Let F be thick tailed and $a_n=[n^p]$, $0<p\leq\frac{1}{2}$. Then $K_n^{(m)}(a)\nrightarrow 1$ a.s. (fails to coverge to 1 a.s.), whenever $\int_{-\infty}^{\infty}((F(x+a)-F(x-a))/(1-F(x))^3)\,dF(x)=\infty$*

Proof of Theorem 2.1. The proof is on the lines of proof of Li (1999). Define $A_n=\{K_n^{(m)}(a)\neq 1\}, n\geq 2$. We show that under the criteria, $P(A_n)\to 0$ and $\sum P(A_n^c\cap A_{n+1})<\infty$ and appeal to the version of the Borel–Cantelli lemma due to Brands et al. (1994), to claim that $K_n^{(m)}(a)\to 1$ a.s., as $n\to\infty$. Observe that $K_n^{(m)}(a)\stackrel{d}{=}K_{a_n}(a)$ implies $P(A_n)=P(K_n^{(m)}(a)\geq 2)=P(K_{a_n}(a)\geq 2)$. Since F is thick tailed, from Pakes and Steutel (1997), we know that $K_n(a)\xrightarrow{p}1$ as $n\to\infty$.

Note that $K_{a_n}(a)\xrightarrow{p}1$, which in turn implies that $P(A_n)\to 0$ as $n\to\infty$. We now establish that $\sum P(A_n^c\cap A_{n+1})<\infty$. Recall that $\eta_n=\max(X_{n-a_n+1},X_{n-a_n+2},\ldots,X_n)$ and note that

$$\begin{aligned}\eta_{n+1}&=\max(X_{(n+1)-a_{n+1}+1},\ldots,X_n,X_{n+1})\\&=\max(X_{n-a_n+1},\ldots,X_n,X_{n+1})\\&=\max(\eta_n,X_{n+1}),\quad \text{if } a_{n+1}=a_n+1\end{aligned}$$

and

$$\eta_{n+1}=\max(X_{n-a_n+2},X_{n-a_n+3},\ldots,X_{n+1})\quad \text{if } a_{n+1}=a_n.$$

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