Journal of Statistical Planning and Inference 140 (2010) 30 – 51

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Statistical inference for reciprocal gamma diffusion process

N.N. Leonenko[a,*], N. Šuvak[b]

[a] Cardiff School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF24 4YH, UK
[b] Department of Mathematics, University of Osijek, Trg Ljudevita Gaja 6, HR-31 000 Osijek, Croatia

A R T I C L E I N F O

Article history:
Received 10 July 2008
Received in revised form
23 February 2009
Accepted 17 June 2009
Available online 21 June 2009

MSC:
60 J60
60 J25
62 M02
62 M05
60 G10
62 M10

Keywords:
Asymptotic normality
Bessel polynomials
Consistency
Heavy-tailed distribution
Martingale estimation equation
Method of moments
Pearson equation
Reciprocal gamma distribution
Reciprocal gamma diffusion
Stationary distribution
Stein equation
Stochastic differential equation

A B S T R A C T

We consider the problem of parameter estimation for an ergodic diffusion with reciprocal gamma invariant distribution. Spectral decomposition of the transition density of such a Markov process is presented in terms of a finite number of discrete eigenfunctions (Bessel polynomials) and eigenfunctions related to a continuous part of the spectrum of the negative infinitesimal generator of reciprocal gamma diffusion. Consistency and asymptotical normality of proposed estimators are presented. Based on the Stein equation for reciprocal gamma diffusion and Bessel polynomials, the hypothesis testing procedure is considered.

1. Introduction

It is now generally accepted that heavy-tailed distributions occur commonly in practice. Their use is now widespread in communication networks, risky asset and insurance modeling. However, the study of stationary processes having these heavy-tailed distributions as their one-dimensional distributions has received rather little attention. Recently, significant improvement in research of Pearson diffusions with heavy tailed marginals is made by Avram et al. (2009). In this paper we focus on such process with reciprocal gamma marginals. This process was first studied by Linetsky (2004a), Bibby et al. (2005) and Heyde and Leonenko (2005) (see also paper by Wong, 1964, and recent paper by Peškir, 2006). In this paper we clarify an important property of reciprocal gamma ergodic diffusion such as the spectral representation of the transition density function obtained

* Corresponding author.
E-mail addresses: LeonenkoN@cardiff.ac.uk (N.N. Leonenko), nsuvak@mathos.hr (N. Šuvak).

0378-3758/$ - see front matter
doi:10.1016/j.jspi.2009.06.009

as the principal solution of the corresponding Fokker–Planck equation. This principal solution is written in form of the finite sum of discrete eigenfunctions (Bessel polynomials) and the integral which is taken over the continuous range of eigenvalues. The general theory can be found in classical books of Karlin and Taylor (1981) and Itô and McKean (1974). The statistical part of the paper contains parametric and semiparametric estimation of parameters of stationary reciprocal gamma diffusion. Statistical inference for ergodic diffusion processes is widely studied (see the recent book by Kutoyants, 2004, the paper by Kutoyants and Yoshida, 2007, and references therein). In particular, the maximum likelihood method and the Bayesian estimation theory are available for the case when diffusion does not depend on an unknown parameter, which is not the case in our situation. Also, the minimum contrast estimation procedure based on Whittle functionals (see Anh et al., 2004, 2007; Leonenko and Sakhno, 2006) is not applicable here, since we are not able to calculate the forth order spectral density. In this paper we use the method of moments and the martingale estimation equation approach. As shown in Kutoyants and Yoshida (2007), asymptotic efficiency of the method of moments can be quite good. Another reason to use the method of moments is that it yields explicit form of estimators for parameters of reciprocal gamma diffusion that correspond to scale and shape parameters of marginal distribution. Since the stationary diffusion process satisfies mixing conditions with the exponentially decaying rate (see Genon-Catalot et al., 2000), using the proper functional central limit theorem and the functional delta method due to Serfling (1980), we are able to prove consistency and asymptotic normality of these estimators. Using the finite system of orthonormal Bessel polynomials we developed a method for calculation of moments of the form $E[X_{s+t}^m X_s^n]$, where m and n are at most equal to the finite number of Bessel polynomials. This method made possible calculation of the explicit form of the limiting covariance matrix of a bivariate estimator of scale and shape parameter of marginal reciprocal gamma distribution (these parameters are also present in drift and diffusion parameter of observed diffusion process). This makes an important problem of constructing asymptotic confidence intervals for unknown parameters operational. A completely different approach to parameter estimation of ergodic diffusions is due to Sørensen and his coauthors. Their results, based on the martingale estimation equation whose solutions are (under some specific assumptions) P-consistent and asymptotically normal estimators of diffusion parameters, are presented in a series of papers. For our work the most important paper is the one by Bibby and Sørensen (1995) in which parameters of the ergodic diffusion process are estimated by the martingale estimation equation taking discretely observations without the assumption that the time between observations tends to zero. Kessler and Sørensen (1999) and Forman and Sørensen (2008) proposed the martingale estimation procedure based on a finite number of eigenfunctions of the negative infinitesimal generator of the corresponding diffusion process (Bessel polynomials in our case). The martingale estimation equation method (see Bibby and Sørensen, 1995) that provides a P-consistent and asymptotically normal estimator is used here for estimation of an unknown autocorrelation parameter. Forman (2005) proposed the least squares method for estimation of an unknown autocorrelation parameter, while Forman and Sørensen (2008) proposed some modification of the Kessler's (2000) estimator. The statistical part also deals with testing reciprocal gamma distributional assumptions and it is based on the Stein equation for reciprocal gamma diffusion. The part concerning the Stein equation is based on general theoretical results presented by Barbour (1990) and Schoutens (2000). Results for testing a statistical hypothesis about reciprocal gamma distribution are based on the results about the Pearson family of distributions of Bontemps and Meddahi (2008). Bontemps and Meddahi (2005) presented important results for testing normality and proved that Hermite polynomials are robust test functions for testing normality using the Stein equation for a standard normal distribution. Our method for testing the hypothesis about reciprocal gamma distribution is based on orthonormal Bessel polynomials, but we were not able to prove their robustness as test functions in the proposed procedure.

2. General information about reciprocal gamma distribution

If random variable Y has gamma distribution with probability density function

$$\mathfrak{g}(x) = \begin{cases} \dfrac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x}, & x > 0, \\ 0, & x \le 0, \end{cases} \tag{2.1}$$

where $\alpha > 0$ is a scale parameter and $\beta > 0$ is a shape parameter, then random variable $X = 1/Y$ has reciprocal or inverse gamma distribution with probability density function

$$\mathfrak{rg}(x) = \begin{cases} \dfrac{\alpha^\beta}{\Gamma(\beta)} x^{-\beta-1} e^{-\alpha/x}, & x > 0, \\ 0, & x \le 0, \end{cases} \tag{2.2}$$

with the same parameters α and β. These distributions will be denoted as follows: $Y \sim \mathfrak{G}(\alpha, \beta)$, $X \sim \mathfrak{RG}(\alpha, \beta)$.

Moment of the kth order of a reciprocal gamma random variable is given by the following expression:

$$E[X^k] = \frac{\alpha^k}{\prod_{i=1}^k (\beta - i)} = \alpha^k \frac{\Gamma(\beta - k)}{\Gamma(\beta)}, \quad \beta > k. \tag{2.3}$$

In particular, expectation and variance of random variable $X \sim \mathfrak{RG}(\alpha, \beta)$ are

$$E[X] = \frac{\alpha}{\beta - 1}, \quad \beta > 1, \quad \mathrm{Var}(X) = \frac{\alpha^2}{(\beta-1)^2(\beta-2)}, \quad \beta > 2.$$