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Testing concerning the homogeneity of some predator–prey populations

Sévérien Nkurunziza*

Department of Mathematics and Statistics, University of Windsor, 401 Sunset Avenue, Windsor, Ontario, Canada N9B 3P4

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ABSTRACT

In this paper, we study an inference problem for a stochastic model where k deterministic Lotka–Volterra systems of ordinary differential equations (ODEs) are perturbed with k pairs of random errors. The k deterministic systems describe the ecological interaction between k predator–prey populations. These k deterministic systems depend on unknown parameters. We consider the testing problem concerning the homogeneity between k pairs of the interaction parameters of the ODEs. We assume that the k pairs of random errors are independent and that, each pair follows correlated Ornstein–Uhlenbeck processes. Thus, we extend the stochastic model suggested in Froda and Colavita [2005. Estimating predator–prey systems via ordinary differential equations with closed orbits. Aust. N.Z. J. Stat. 2, 235–254] as well as in Froda and Nkurunziza [2007. Prediction of predator–prey populations modeled by perturbed ODE. J. Math. Biol. 54, 407–451] where $k=1$. Under this statistical model, we propose a likelihood ratio test and study the asymptotic properties of this test. Finally, we highlight the performance of our method through some simulations studies.

1. Introduction

Consider k different geographical areas numbered from 1 to k, $(k \geq 1)$. In geographical area number m, we consider the predator–prey system described by the Lotka–Volterra ODE system (Lotka, 1925; Volterra, 1931)

$$dx_m(t)/dt = (\eta_m - \beta_m y_m(t))x_m(t), \quad dy_m(t)/dt = (\gamma_m x_m(t) - \delta_m)y_m(t), \tag{1}$$

with $(x_m(0), y_m(0)) = (x_{m0}, y_{m0})$ fixed, $x_{m0} > 0$, $y_{m0} > 0$, $\eta_m > 0$, $\beta_m > 0$, $\gamma_m > 0$, $\delta_m > 0$, $m = 1, 2, \ldots, k$. Also, we suppose that each initial value (x_{m0}, y_{m0}) is different from an equilibrium point. Hence, the solution of the system (1) cannot be trivial.

Theoretically, $x_m(t)$ and $y_m(t)$ are the population sizes in the area m (at time t) of the prey and the predator, respectively. The parameter η_m is the birth rate of the prey when the predator is absent, δ_m is the death rate of the predator when the prey is absent while β_m and γ_m are the interaction parameters. The parameters η_m and δ_m are considered as constant and intrinsic to the species of the prey and the predator respectively. In practice, for each area number m, we have N pairs of observations $(X_1^{(m)}, Y_1^{(m)}), (X_2^{(m)}, Y_2^{(m)}), \ldots, (X_N^{(m)}, Y_N^{(m)})$ observed at discrete times $0 < t_1 < t_2 < \cdots < t_N$, where $X_i^{(m)}$ and $Y_i^{(m)}$, represent respectively the sizes of the population of the prey and the predator observed at time t_i, $i = 1, 2, \ldots, N$. We assume that these N pairs of observations are generated by a stochastic model which is given in Section 2.

So far, Froda and Colavita (2005) and Nkurunziza (2005) as well as Froda and Nkurunziza (2007) proposed estimation methods when $k=1$, for the four parameters η_m, β_m, γ_m, δ_m. Also, for $k=1$, Nkurunziza (2008) proposed the likelihood ratio test for the two interaction parameters γ_m and β_m. In this paper, we are interested in testing problems concerning the homogeneity of the k pairs of interaction parameters (β_m, γ_m). In order to simplify the problem, we assume that the k pairs of parameters (η_m, δ_m) are

* Tel.: +1 519 253 3000x3017; fax: +1 519 971 3649.
E-mail address: severien@uwindsor.ca.

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equal and do not depend on the interaction parameters. This assumption is justified by the fact that η_m and δ_m are considered as constant and intrinsic to the species of the prey and the predator, respectively. In other words, we assume that the factor geographical area may affect the interaction parameters, but cannot affect the intrinsic parameters. To simplify the notation, let (η, δ) be the common value of (η_m, δ_m). Statistically, we are interested in testing

$$H_0 : (\gamma_1, \beta_1) = (\gamma_2, \beta_2) = \cdots = (\gamma_k, \beta_k) \text{ against } H_A : (\gamma_m, \beta_m) \neq (\gamma_j, \beta_j) \text{ for some } 1 \leq m < j \leq k. \tag{2}$$

The major difficulty of this problem, resides in the fact that the systems (1) do not admit an explicit analytical solution. In other words, we cannot write the solution $(x_m(t), y_m(t))$ as an explicit function of the four parameters $(\gamma_m, \beta_m, \delta, \eta)$. Because of that, there is no straightforward solution to the testing problem (2). To this end, we use the fact that the trajectory $(x_m(t), y_m(t))$ is a periodic function.

Indeed, let $\varpi(\gamma_m, \beta_m, \delta, \eta, x_{m0}, y_{m0})$ be the trajectory period. Thus, the period $\varpi(\gamma_m, \beta_m, \delta, \eta, x_{m0}, y_{m0})$ is a function of $(\gamma_m, \beta_m, \delta, \eta, x_{m0}, y_{m0})$. It should be noted that, when the initial value is close to the equilibrium point, $(\delta/\gamma_m, \eta/\beta_m)$, the period $\varpi(\gamma_m, \beta_m, \delta, \eta, x_{m0}, y_{m0}) \approx 2\pi/\sqrt{\delta\eta}$ and this last term does not depend on the parameters γ_m, β_m. In this paper, we give a sufficient condition, on (x_{m0}, y_{m0}), for the function $\varpi(\gamma_m, \beta_m, \delta, \eta, x_{m0}, y_{m0})$ to be constant with respect to the parameters γ_m, β_m (see Corollary A.2 in Appendix A). For more details about the intrinsic property of the period for some predator–prey systems, we refer the reader to Ginzburg and Colyvan (2004, p. 64, 77).

The rest of the paper is organized as follows. In Section 2, we give the preliminary results and set up assumptions as well as some notation. Section 3 presents the likelihood ratio test when the nuisance parameters are known. In Section 4, we present the test when these parameters are unknown and simulation studies. Section 5 gives the Conclusion. Technical details and proofs are relegated to Appendix A and Appendix B.

2. Statistical model and preliminary results

In this section, we present the statistical model and set up some notation used in the sequel. As mentioned in the Introduction, we would like to test

$$H_0 : (\gamma_1, \beta_1) = (\gamma_2, \beta_2) = \cdots = (\gamma_k, \beta_k) \text{ against } H_A : (\gamma_m, \beta_m) \neq (\gamma_j, \beta_j) \text{ for some } 1 \leq m < j \leq k. \tag{3}$$

Further, we have other nuisance parameters like as $\alpha, \delta, x_{m0}, y_{m0}$. The solution of the testing problem (3) is given in two main steps. First, we consider the case where we know the nuisance parameters. In this case, the proposed test is the likelihood ratio test. Second, we consider the more realistic case where the nuisance parameters are unknown. In this last case, we establish a test which is asymptotically as powerful as the likelihood ratio test.

From the methodological point of view, we adopt a statistical model that is similar to that given in Froda and Nkurunziza (2007) for the particular case where $k=1$. Namely, we assume that, $(X_i^{(m)}, Y_i^{(m)}), i=1,2,\ldots,N, m=1,2,\ldots,k$ are observed at discrete times $0 < t_1 < t_2 < \cdots < t_N$; where $X_i^{(m)} \equiv X_m(t_i)$, $Y_i^{(m)} \equiv Y_m(t_i)$, $m = 1, 2, \ldots, k$, and, for each geographical area $m = 1, 2, \ldots, k$, the observed values are generated by a process with continuous paths $\{(X_m(t), Y_m(t)), 0 \leq t \leq T\}$, satisfying

$$\log X_m(t) = \log x_m(t) + e_t^{m,X}, \quad \log Y_m(t) = \log y_m(t) + e_t^{m,Y}, \quad m = 1, 2, \ldots, k. \tag{4}$$

Also, we assume that, the k pairs of process errors are independent and that, for each geographical area $m = 1, 2, \ldots, k$, each noise component $\{(e_t^{m,X}, e_t^{m,Y}), 0 \leq t \leq T\}$ is an Ornstein–Uhlenbeck process (see e.g. Kutoyants, 2004, p. 51), with a particular dependence structure defined in Assumption $(\mathscr{C}_1)$. More precisely,

$$de_t^{m,X} = -c_m e_t^{m,X}\, dt + \tau_m\, dW_t^{m,X}, \quad de_t^{m,Y} = -c_m e_t^{m,Y}\, dt + \tau_m\, dW_t^{m,Y}, \tag{5}$$

where $\tau_m > 0$, $c_m > 0$, and $\{W_t^{m,X}, t \geq 0\}$ as well as $\{W_t^{m,Y}, t \geq 0\}$ are Wiener processes which satisfy the following assumption.

Assumption $(\mathscr{C}_1)$**.** For $m = 1, 2, \ldots, k$, the Wiener processes $\{W_t^{m,X}, t \geq 0\}$ and $\{W_t^{m,Y}, t \geq 0\}$ are jointly Gaussian and such that, for all $i, j = 1, 2, 3, \ldots$, $\mathrm{Cov}(W_{t_i}^{m,X}, W_{t_j}^{m,X}) = \rho_m \min(t_i, t_j)$, where $|\rho_m| < 1$.

Proposition 1 gives necessary and sufficient conditions for Assumption $(\mathscr{C}_1)$ to be satisfied. Moreover, this proposition highlights the existence of Wiener processes satisfying $(\mathscr{C}_1)$. Also, under Assumption $(\mathscr{C}_2)$, Proposition 1, allows us to simplify some computations. Indeed, in the sequel, we assume, without loss of generality, that $\rho_m = 0, m = 1, 2, \ldots, k$. In fact, if $\rho_m \neq 0$ but $|\rho_m| < 1$, we can consider the following transformation:

$$\tilde{e}_t^{m,X} = (e_t^{m,X} + e_t^{m,Y})/\sqrt{2(1+\rho_m)}, \quad \tilde{e}_t^{m,Y} = (e_t^{m,Y} - e_t^{m,X})/\sqrt{2(1-\rho_m)}. \tag{6}$$

Proposition 1 shows that, $\{\tilde{e}_t^{m,X}, t \geq 0\}$ and $\{\tilde{e}_t^{m,Y}, t \geq 0\}$ are independent Ornstein–Uhlenbeck processes (i.e. satisfy $(\mathscr{C}_1)$ where ρ_m transforms into $\tilde{\rho}_m = 0$).