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# Designs for first-order interactions in paired comparison experiments with two-level factors

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ABSTRACT

For paired comparison experiments involving options described by a common set of two-level factors a new method for generating exact designs is presented. These designs allow the efficient estimation of main effects and first-order interactions and perform better than alternative designs available in the literature.



## 1. Introduction

Paired comparison experiments aim to help understand how preferences for goods or services are influenced by the features of competing options. To this end respondents are asked to evaluate pairs of options, where typically each option is represented by a combination of the levels of several factors and an experimental design is used to generate the pairs. The resulting data are then used to estimate utility values which reflect the influence of the factors.

This paper presents a new method for constructing efficient designs for paired comparison experiments in which the options are characterized by a common set of two-level factors and both main effects and first-order interactions are to be estimated while all higher-order interactions are assumed to be negligible. Optimal designs for estimating main effects and first-order interactions in paired comparison experiments when the factors are continuous with levels between $-1$ and 1 have been derived by van Berkum (1987, pp. 30–31). These designs remain optimal for two-level factors and were also obtained by Street et al. (2001) using a different method. Corresponding results when the common number of factor levels is larger than two are given by Graßhoff et al. (2003).

Generally, the optimal designs of van Berkum (1987) and Street et al. (2001) are too large for most applications and smaller alternatives with good efficiency properties have been presented by Street and Burgess (2004). In what follows, we describe a new class of designs. For the same number of pairs these designs are equally or more efficient than the ones given by Street and Burgess (2004). Moreover, several comparatively small and efficient designs are presented for which no corresponding designs generated by the method of Street and Burgess (2004) are available in the literature.

## 2. Known optimality results

Suppose there are $K$ two-level factors that are assumed to drive the preferences for the options in a paired comparison experiment. Without loss of generality we use the numbers 1 and $-1$ to represent the first and second levels of each

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factor, respectively. The first option in each pair is denoted by $\mathbf{s}=(s_1,\ldots,s_K)$ and the second by $\mathbf{t}=(t_1,\ldots,t_K)$, which are both elements of the set $\{-1,1\}^K$.

We assume that for every pair $(\mathbf{s},\mathbf{t})$ the comparison of $\mathbf{s}$ and $\mathbf{t}$ results in a numerical value $Y(\mathbf{s},\mathbf{t})$ which is essentially the difference between the utility associated with $\mathbf{s}$ and $\mathbf{t}$, respectively. This type of response occurs naturally, for example, in widely used variants of conjoint analysis (Green and Srinivasan, 1990) where options are presented in pairs and responses are collected as ratings. More formally, we consider a general linear model

$$Y(\mathbf{s},\mathbf{t})=(\mathbf{f}(\mathbf{s})-\mathbf{f}(\mathbf{t}))^\top\boldsymbol{\beta}+\varepsilon, \tag{1}$$

where $\mathbf{f}$ is a vector of known regression functions, $\boldsymbol{\beta}$ is a vector of unknown parameters and the random errors $\varepsilon$ associated with different pairs are assumed to be uncorrelated with constant variance. In this paper, the influence of the two-level factors is described as in the regression version of a standard analysis of variance model with effects-coded explanatory variables where for every $\mathbf{x}=(x_1,\ldots,x_K)\in\{-1,1\}^K$ the vector $\mathbf{f}(\mathbf{x})$ is given by

$$\mathbf{f}(\mathbf{x})=(x_1,\ldots,x_K,x_1x_2,\ldots,x_1x_K,x_2x_3,\ldots,x_{K-1}x_K)^\top. \tag{2}$$

Let $\mathbf{X}=(\mathbf{f}(\mathbf{s}_1)-\mathbf{f}(\mathbf{t}_1),\ldots,\mathbf{f}(\mathbf{s}_N)-\mathbf{f}(\mathbf{t}_N))^\top$ be the design matrix in model (1) for an exact design $\xi_N$ of size $N$ consisting of the pairs $(\mathbf{s}_1,\mathbf{t}_1),\ldots,(\mathbf{s}_N,\mathbf{t}_N)$. The quality of $\xi_N$ is reflected by its normalized information matrix $\mathbf{M}(\xi_N)=(1/N)\mathbf{X}^\top\mathbf{X}=(1/N)\sum_{n=1}^N(\mathbf{f}(\mathbf{s}_n)-\mathbf{f}(\mathbf{t}_n))(\mathbf{f}(\mathbf{s}_n)-\mathbf{f}(\mathbf{t}_n))^\top$ which is proportional to the inverse of the covariance matrix of the least squares estimator for $\boldsymbol{\beta}$.

More generally, we consider approximate designs which are defined as probability measures on the design region $\mathcal{X}=\{-1,1\}^K\times\{-1,1\}^K$ of all pairs $(\mathbf{s},\mathbf{t})$. Every exact design $\xi_N$ consisting of $N$ pairs $(\mathbf{s}_1,\mathbf{t}_1),\ldots,(\mathbf{s}_N,\mathbf{t}_N)$ can be identified with the approximate design $\tilde{\xi}_N$ which assigns weight $\tilde{\xi}_N(\mathbf{s},\mathbf{t})=|\{n\in\{1,\ldots,N\}:(\mathbf{s}_n,\mathbf{t}_n)=(\mathbf{s},\mathbf{t})\}|/N$ to the pair $(\mathbf{s},\mathbf{t})$ in $\mathcal{X}$. Conversely, every approximate design $\xi$ which assigns only rational weights $\xi(\mathbf{s},\mathbf{t})$ to all pairs $(\mathbf{s},\mathbf{t})$ in its support can be realized as an exact design $\xi_N$ for some $N$. The information matrix of an approximate design $\xi$ in the linear paired comparison model (1) is defined by

$$\mathbf{M}(\xi)=\sum_{(s,t)\in\mathcal{X}}\xi(\mathbf{s},\mathbf{t})(\mathbf{f}(\mathbf{s})-\mathbf{f}(\mathbf{t}))(\mathbf{f}(\mathbf{s})-\mathbf{f}(\mathbf{t}))^\top.$$

Note that for an exact design $\xi_N$ the normalized information matrix $\mathbf{M}(\xi_N)$ coincides with the information matrix $\mathbf{M}(\tilde{\xi}_N)$ of the corresponding approximate design $\tilde{\xi}_N$.

An approximate design $\xi^*$ is $D$-optimal if it maximizes the determinant of the information matrix, that is, if $\det\mathbf{M}(\xi^*)\geq\det\mathbf{M}(\xi)$ for every approximate design $\xi$ on $\mathcal{X}$. In general, the quality of an approximate design $\xi$ can be assessed by means of the $D$-efficiency $\mathrm{eff}_D(\xi)=(\det\mathbf{M}(\xi)/\det\mathbf{M}(\xi^*))^{1/p}$ where $\xi^*$ is $D$-optimal and $p=K+K(K-1)/2$ is the number of model parameters.

The design region $\mathcal{X}$ can be partitioned into disjoint sets such that the pairs in each set differ only in some of the factors. More precisely, for $d=0,\ldots,K$ let $\mathcal{X}_d=\{(\mathbf{s},\mathbf{t})\in\mathcal{X}:|\{k:s_k\neq t_k\}|=d\}$ be the set of the $N_d=2^KK!/[d!(K-d)!]$ pairs which vary in exactly $d$ factors and denote by $\overline{\xi}_d$ the uniform approximate design which gives equal weight $\overline{\xi}_d(\mathbf{s},\mathbf{t})=1/N_d$ to each pair in $\mathcal{X}_d$ and weight zero to all remaining pairs in $\mathcal{X}$. Following Graßhoff et al. (2003) we refer to $d$ as the comparison depth.

It was shown by van Berkum (1987, p. 30) that the information matrix of $\overline{\xi}_d$ is equal to

$$\mathbf{M}(\overline{\xi}_d)=\begin{pmatrix}\frac{4d}{K}\mathbf{I}_K & \mathbf{0}\\ \mathbf{0} & \frac{8d(K-d)}{K(K-1)}\mathbf{I}_{K(K-1)/2}\end{pmatrix}, \tag{3}$$

where $\mathbf{I}_m$ is the identity matrix of order $m$ for every $m$. Moreover, if the number of factors $K$ is odd, then the design $\xi^*=\overline{\xi}_{d^*}$ with information matrix $\mathbf{M}(\xi^*)=4d^*/K\mathbf{I}_p$ is $D$-optimal where $d^*=(K+1)/2$. If $K$ is even, then a convex combination $\xi^*=w^*\overline{\xi}_{d^*}+(1-w^*)\overline{\xi}_{d^*+1}$ where $d^*=K/2$ and $w^*=(d^*+1)/(K+1)$ is $D$-optimal with information matrix $\mathbf{M}(\xi^*)=4(d^*+1)/(K+1)\mathbf{I}_p$.

Although $\overline{\xi}_{d^*}$ is not $D$-optimal for even numbers of factors, it is nearly so with $\mathrm{eff}_D(\overline{\xi}_{d^*})>0.99$ if $K>2$. Moreover, uniform designs $\overline{\xi}_d$ where $d=d^*-1$ or $d=d^*+1$ are often highly efficient. Table 1 presents corresponding $D$-efficiencies of the uniform designs with comparison depths $d^*-1$, $d^*$ and $d^*+1$ for $K=3,\ldots,10$.

Despite their good efficiency properties these designs are usually much too large to be used in practice. In the next section we therefore show how highly efficient, but smaller, exact designs $\xi_{N,d}$ where $d=d^*$ or $d=d^*+1$ can be generated

**Table 1**
$D$-efficiencies of uniform designs for estimating main effects and interactions.

| $K$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| $d^*$ | 2 | 2 | 3 | 3 | 4 | 4 | 5 | 5 |
| $\mathrm{eff}_D(\overline{\xi}_{d^*})$ | 1.0000 | 0.9903 | 1.0000 | 0.9967 | 1.0000 | 0.9985 | 1.0000 | 0.9992 |
| $\mathrm{eff}_D(\overline{\xi}_{d^*-1})$ | 0.7071 | 0.6315 | 0.8736 | 0.8161 | 0.9306 | 0.8908 | 0.9564 | 0.9279 |
| $\mathrm{eff}_D(\overline{\xi}_{d^*+1})$ | 0.0000 | 0.9801 | 0.8399 | 0.9948 | 0.9222 | 0.9979 | 0.9533 | 0.9989 |